Certain Hecke Algebras of $\mathbf{GL}_n(\mathbb{Q}_p)$

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Abstract

We give an overview of various results related to Hecke algebras of $GL_n(\mathbb{Q}_p)$. Our primary aim is to provide a short, and nearly self contained proof of the structure of the Iwahori Hecke algebra. This contains a proof new to the author of a well known result. Our secondary aim is to inspect "Congruence Hecke algebras" studied in Chapter 3, Section 2 of [How85]. We calculate their 1-dimensional representations, and improve on bounds of [Ber74].

Contents

Chapter 1

Introduction

The various Hecke algebras of $\mathrm{GL}_n(\mathbb{Q}_p)$ are host to a wealth of beautiful mathematics. Their structure is intimately related to the data of admissible representations of $\mathrm{GL}_n(\mathbb{Q}_p)$. This is a topic of considerable interest as related to the Langlands Program, where they play a crucial role in the Local Langlands Correspondence.

In this paper, we aim to give an overview of these results, with a focus on those that bound the dimensions of their irreducible modules. One of our main aims is to give a short and nearly self contained proof of the structure of the Iwahori-Hecke algebra of $\mathrm{GL}_n(\mathbb{Q}_p)$.

In Chapter 1, we introduce the p-adic numbers and notions important to the representation theory of $\mathrm{GL}_n(\mathbb{Q}_p)$. In Chapter 2, we define our Hecke algebras and utilise results of [BK98] and [BS17] to show that their simple modules contain the data of admissible representations of $\mathrm{GL}_n(\mathbb{Q}_p)$. Moreover, we briefly study the Spherical Hecke algebras of $\mathrm{GL}_n(\mathbb{Q}_p)$.

In Chapter 3, we discuss affine reflection groups, in order to prove a crucial Proposition of [IM65] regarding the Iwahori subgroup of $\mathrm{GL}_n(\mathbb{Q}_p)$. In Chapter 4, we follow [IM65] and utilise this Proposition to prove the Iwahori-Bruhat Decomposition of $\mathrm{GL}_n(\mathbb{Q}_p)$, allowing us to understand the Iwahori Hecke Algebra. This includes a proof new to the author of an enjoyable Proposition (Proposition 4.6).

After this, we outline and specialize the methods of [Lus89] to bound the dimension of irreducible modules of the Iwahori Hecke Algebra at $n = 2$. Finally, in Chapter 5, we move on to algebras studied by Howe and Moy in [How85], and calculate their 1-dimensional modules, as well as improve bounds of Bernshtein in [Ber74] on the dimension of their irreducible modules. Our calculations give a criterion, in one case improving the bound by roughly a factor of p^4 and highlighting an ideal acting by 0, and in the other improving by a square root.

1.1 The p-adic Numbers

The p-adic numbers are a completion of $\mathbb{Q}$ with respect to a non-standard metric. We define it as follows.

Definition 1.1. Fix $p \in \mathbb{N}$ prime. For $x \in \mathbb{Q}$, we may uniquely write $x = p^n \cdot \frac{a}{b}$, with $a, b \in \mathbb{Z}$, co-prime to p and each other, and $n \in \mathbb{Z}$.

Define $|\cdot|_p : \mathbb{Q} \to \mathbb{R}$, via

$$|x|_p = p^{-n}$$

This is obviously a norm, which in fact obeys a condition stronger than the triangle inequality, namely: $|x + y|_p \leq \max(|x|_p, |y|_p)$.

We write the completion of $\mathbb{Z}$ (resp. $\mathbb{Q}$) with this norm as $\mathbb{Z}_p$ (resp. $\mathbb{Q}_p$) , whose elements we call *p-adic integers* (resp. *p-adic numbers*).

Remark 1.2. Much of what we will do is true in the more general setting of a Non-Archimedean Local Field. These are finite extensions of either $\mathbb{F}_p((t))$ (the field of Laurent series over a finite field) or of $\mathbb{Q}_p$ (see Theorem 6.15 of [All16]). We will restrict ourselves to the p-adic numbers.

Proposition 1.3. $\mathbb{Q}_p$ is a field.

This is left as an exercise to the reader[1].

Example 1.4. Every $z \in \mathbb{Z}_p$ may be written as

$$z = \sum_{n=0}^{\infty} a_n p^n$$

where $a_n \in \{0, \ldots, p-1\}$ for each n. Note, further that this expansion is unique.

Example 1.5. Every $z \in \mathbb{Q}_p$ may be written

$$z = \sum_{n=-k}^{\infty} a_n p^n$$

for a_n as above and $k \in \mathbb{N}$.

Notation 1.6. Owing to its frequent usage, we will write $\mathbb{Z}/p^n$ for $\mathbb{Z}/p^n\mathbb{Z}$.

[1] The following example and Proposition 5.5 may be useful

Remark 1.7. Those with a categorical bent will be excited that the p-adics can in fact be expressed as

$$\mathbb{Z}_p = \varprojlim_{n \in \mathbb{N}} (\mathbb{Z}/p^n)$$

$$\mathbb{Q}_p = \mathrm{Frac}(\mathbb{Z}_p)$$

Because our field $\mathbb{Q}_p$ was defined by way of a metric, it has a natural topology.

Lemma 1.8. *$\mathbb{Z}_p$ is compact and open in $\mathbb{Q}_p$.*

Proof. Left to the reader. □

We now define ring homomorphisms $\pi_n : \mathbb{Z}_p \to \mathbb{Z}/p^n$. By uniqueness of our expansion, define[2]:

$$\pi_n(a_0 + a_1 p + a_2 p^2 + \ldots) = a_0 + a_1 p + \ldots + a_{n-1} p^{n-1} (\mathrm{mod}\ p^n)$$

Lemma 1.9. *$\pi_n : \mathbb{Z}_p \to \mathbb{Z}/p^n$ is continuous, where we endow $\mathbb{Z}/p^n$ with the discrete topology*

Proof. It is easy to check that our fibres are open balls. □

1.2 Representations of GL_n

In this Section, we lay the groundwork to state a result (Theorem 2.1) motivating our interest in our objects of study. All of our representations will be complex.

We could at this point, specialise to $\mathrm{GL}_n(\mathbb{Q}_p)$. We will refrain from doing so to retain generality, and to highlight some of the most useful features of $\mathrm{GL}_n(\mathbb{Q}_p)$. For this we define a new class of groups.

Definition 1.10. Let G be a group, and a topological space. We call G:

1. a *topological* group when multiplication and inversion are continuous maps.
2. a *t.d.* group when every neighbourhood of the identity has a compact open subgroup.

The latter definition may be found in [Car79]. As it is our focus, we should certainly hope $\mathrm{GL}_n(\mathbb{Q}_p)$ fits into our definition. Our next example verifies this, and contains the definition of an important map.

[2] cf. Remark 1.7.

Example 1.11 ($\mathrm{GL}_n(\mathbb{Q}_p)$ is *t.d.*)**.** The topology on $\mathrm{GL}_n(\mathbb{Q}_p)$ is the obvious one inherited from $\mathbb{Q}_p^{n^2}$. Moreover, multiplication is obviously continuous as it is polynomial in the matrix entries, and inversion is continuous by Cramer's rule. For the *t.d.* property, we introduce the following subgroups:

$$\begin{aligned} K_0 &= \mathrm{GL}_n(\mathbb{Z}_p) = q_1^{-1}(\mathrm{GL}_n(\mathbb{F}_p)) \\ K_m &= q_m^{-1}(\mathrm{Id}_n) \end{aligned}$$

Where we now define $q_m : \mathrm{Mat}_n(\mathbb{Z}_p) \to \mathrm{Mat}_n(\mathbb{Z}/p^n)$ coordinate-wise by π_m. Our first equality is not obvious, though can be seen via the use of von-Neumann series.

Lemma 1.12. *q_m is continuous, and a group homomorphism*

Proof. Taking $M = (m_{ij}) \in \mathrm{Mat}_n(\mathbb{Z}/p^n)$

$$q_m^{-1}(M) = \bigcap_{i,j=1,\dots,n} \{A \in \mathrm{Mat}_n(\mathbb{Z}_p) : \pi_m(a_{ij}) = m_{ij}\}$$

We note

$$\{A \in \mathrm{Mat}_n(\mathbb{Z}_p) : \pi_m(a_{ij}) = m_{ij}\}$$

is a closed subset of $\mathrm{Mat}_n(\mathbb{Z}_p)$ by continuity of π_m, as it is a coordinate-wise product of closed sets. So then the sets in our intersection are closed. So clearly $q_m^{-1}(M)$ is closed. □

Continuity of this map shows K_i is both open and closed in $\mathrm{Mat}_n(\mathbb{Z}_p)$ for each $i \in \mathbb{N}_0$, which also gives compactness by Lemma 1.8. As $\mathrm{Mat}_n(\mathbb{Z}_p)$ is both open and closed in $\mathrm{Mat}_n(\mathbb{Q}_p)$, we inherit the same result for $\mathrm{GL}_n(\mathbb{Q}_p)$. From here, the *t.d.* property is easily seen.

Definition 1.13. For a representation (π, V) of a *t.d.* group G, and $H \subset G$ a subgroup, we write

$$V^H = \{v \in V : \pi(h) \cdot v = v \text{ for all } h \in H\},$$

and we call V^H the H *fixed point space* of V.

Definition 1.14. A representation (π, V) of a *t.d.* group G is

(i) *smooth* if the stabilizer of any $v \in V$ is open in G.

(ii) *admissible* when it is smooth, and V^K is finite dimensional for any compact subgroup $K \subset G$.

The notion of admissibility was introduced by Jacquet and Langlands in [JL69], in which the Jacquet-Langlands correspondence was proven. We have one final definition for this Chapter:

Definition 1.15. Write $\mathfrak{R}(G)$ for the category of smooth, complex representations of G. Fixing a compact, open subgroup K of G, denote by $\mathfrak{R}_K(G)$ the full subcategory of representations generated by their K fixed point space.

Remark 1.16. This second definition may seem slightly bizarre. Notably, all irreducible representations of G with a non-zero K fixed point space are in $\mathfrak{R}_K(G)$.

We have now built up the definitions on the representation theory side, and can move on to define our algebras. We finish off the Chapter with a calculation.

Example 1.17. Take a smooth finite dimensional irreducible representation (π, V) of $\mathbb{Q}_p^\times$. Of course, by Schur's lemma, we have that V is 1 dimensional, so that we may write $V = \mathrm{Sp}(v)$ and then further, the stabiliser of v must act trivially. Of course, this is open around the identity, and so contains some subgroup $K := 1 + p^n\mathbb{Z}_p$. Notably $K = \ker(\pi_n|_{\mathbb{Z}_p^\times})$, giving $\mathbb{Z}_p^\times/K \cong (\mathbb{Z}/p^n)^\times$. V of course then descends to a representation of $\mathbb{Q}_p^\times/K$. We have

$$K \triangleleft \mathbb{Z}_p^\times \triangleleft \mathbb{Q}_p^\times,$$

and so of course, an isomorphism theorem tells us that

$$(\mathbb{Q}_p^\times/K)/(\mathbb{Z}_p^\times/K) \cong \mathbb{Q}_p^\times/\mathbb{Z}_p^\times \cong \mathbb{Z},$$

so that we get a short exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}_p^\times/K & \longrightarrow & \mathbb{Q}_p^\times/K & \longrightarrow & \mathbb{Q}_p^\times/\mathbb{Z}_p^\times & \longrightarrow & 0 \\ & & \downarrow \wr & & \| & & \downarrow \wr & & \\ 0 & \longrightarrow & (\mathbb{Z}/p^n)^\times & \longrightarrow & \mathbb{Q}_p^\times/K & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \end{array}$$

As $\mathbb{Z}$ is projective[3] we have

$$\mathbb{Q}_p^\times/K \cong \mathbb{Z} \times (\mathbb{Z}/p^n)^\times$$

whose 1-dimensional representations are easily understood. Indeed, when $p \neq 2$, both of these groups are cyclic (see Theorem 4 of [Dal18]).

[3] If the reader is not familiar with this notion, this sentence might be replaced with "any SES of Abelian groups with $\mathbb{Z}$ in the third place splits".

Chapter 2

Convolution Algebras

In this Chapter we study Convolution algebras and specialize to the Hecke algebra $\mathcal{H}(G//K)$ of a pair (G, K), for G a *t.d.* group, and a compact open subgroup K. We finish off by producing a bound on the dimensions of irreducible modules of $\mathcal{H}(G//K_0)$. For the rest of the Chapter, G is a *t.d.* group.

As we have already laid out half of the theory, to motivate the coming Section, we spoil the theorem teased in the last Chapter:

Theorem 2.1. *Suppose $K \subset GL_n(\mathbb{Q}_p)$ is a Moy-Prasad subgroup. Then the functor*

$$M : \mathfrak{R}_K(GL_n(\mathbb{Q}_p)) \to \mathcal{H}(GL_n(\mathbb{Q}_p)//K)\text{-}mod$$

$$V \mapsto V^K$$

defines an equivalence of categories.

Remark 2.2.

1. Further to this theorem, for any compact open subgroup K of $\mathrm{GL}_n(\mathbb{Q}_p)$, should V, W be admissible representations of G with $V^K \cong W^K$ as $\mathcal{H}(G//K)$-modules, then $V \cong W$ (see Theorem 1 of [Bum10]). However, this does not define necessarily an equivalence of categories. An example of this failure is given in [BK98] (see the final remark of Section 2). We will use this to get around that some subgroups we consider are not Moy-Prasad.

2. As we have not yet defined most of these terms, we will not explain the utility of this theorem until slightly later.

2.1 Definition

We build our Hecke algebras from a "convolution" product on the functions on G. We take inspiration from the usual convolution product on the real line:

Example 2.3. For $f, g \in C_C((\mathbb{R}, +))$[1] this is

$$(f \star g)(x) = \int_{\mathbb{R}} f(y)g(x-y)dy$$

with the usual Lebesgue measure.

Of course, to implement this in our case, we require a measure on G. Note that in fact, any *t.d.* group is locally compact (i.e every point has a compact neighbourhood). The identity has an open compact neighbourhood and left multiplication by any element $x \in G$ is a homeomorphism. This maps e to x and preserves openness and compactness.

Fact 2.4 (Existence of the Haar Measure)**.** Let G be a locally compact topological group. There exists a unique (up to multiplicative constant) countably additive, non-zero measure μ on G such that:

(i) $\mu(S) = \mu(gS)$ for all Borel[2] sets $S \subset G$,

(ii) $\mu(K) < \infty$ for all compact subsets $K \subset G$,

(iii) $\mu(S) = \inf\{\mu(U) : U \text{ open}, S \subset U\}$ for $S \subset G$ Borel,

(iv) $\mu(U) = \sup\{\mu(K) : K \text{ compact}, U \subset K\}$ for $U \subset G$ open.

A proof of this may be found in [Coh13] as Theorem 9.2.2. Property (i) is the most crucial, which we will refer to as *translation invariance.*

Example 2.5. The Haar Measure on $(\mathbb{Q}_p, +)$ has, up to a constant factor

$$\mu^+(p^n\mathbb{Z}_p) = p^{-n}.$$

Example 2.6. The Haar Measure on $(\mathbb{Q}_p, +)$ is given up to a constant factor by

$$\mu^\times(E) = \int_{\mathbb{Q}_p} \frac{1}{|x|_p} d\mu^+(x).$$

[1]This is defined below

[2]A Borel set is a member of the σ-algebra generated by open sets of G

For the most part, we will ignore the particulars of our measures, in favour of appealing to their properties.

Definition 2.7. We write $C_C(G)$ for the complex valued, continuous, locally constant, compactly supported functions on G.

Which we give the following product:

Definition 2.8. For $f, g \in C_C(G)$ define

$$(f \star g)(x) = \int_G f(h)g(h^{-1}x)dh.$$

It is clear that our product is bilinear, we need to check associativity.

Proposition 2.9. $(\cdot \star \cdot) : C_C(G) \times C_C(G) \to C_C(G)$ is associative

Proof. We use Fubini's theorem (which holds due to compactness) and translation invariance. This calculation is preserved so the reader has seen its flavour, but we will omit them in the future. Further, that the product lands in $C_C(G)$ is left to the reader.

$$\begin{aligned}
(f \star (g \star h))(a) &= \int_G f(x)(g \star h)(x^{-1}a)dx \\
&= \int_G f(x) \int_G g(y)h(y^{-1}x^{-1}a)dydx \\
&= \int_G f(x) \int_G g(x^{-1}u)h(u^{-1}a)dudx \\
&= \int_G h(u^{-1}a) \int_G f(x)g(x^{-1}u)dxdu \\
&= \int_G (f \star g)(u)h(u^{-1}a)dxdu \\
&= ((f \star g) \star h)(a).
\end{aligned}$$

□

Definition 2.10. We will write $\mathcal{H}(G)$ to denote the algebra $(C_C(G), \star)$.

2.2 Hecke Algebras

It is now time to define our central object of study, the Hecke algebra of a pair (G, K). We proceed as follows:

Definition 2.11. Let G be a *t.d.* group, and K be an open, compact subgroup. Define

$$\mathcal{H}(G//K) := \{f \in C_C(G) : f(k_1 g k_2) = f(g) \text{ for all } k_1, k_2 \in K, \text{ and } g \in G\}$$

which we call the *Hecke algebra of the pair (G,K)*. In words, these are elements of $C_C(G)$ invariant under right and left translation by K. We call these the K *bi-invariant functions* on G.

Now, for such (G, K), it is a quick direct calculation to show that this is in fact an algebra. In order to aid in the proof of our theorem, we will take a slightly more roundabout approach:

Proposition 2.12. Suppose the Haar measure is normalized so that $\mathrm{Vol}(K) = 1$. Define $e_K = \mathbb{1}_K$. Then, for $f \in \mathcal{H}(G)$, and $x \in G$ we have

(i) $(e_K \star f)(x) = \int_K f(y^{-1}x)dy$

(ii) $(f \star e_K)(x) = \int_K f(xy)dy$

(iii) $e_K \star f \star e_K \in \mathcal{H}(G//K)$.

(iv) e_K acts as the identity on $\mathcal{H}(G//K)$

Proof. These are immediate from the definitions, and from translation invariance. For (iii) $f \star e_K$ is right invariant by K, and $e_K \star f$ is left invariant, so associativity gives the result. (iv) is immediate from (i) and (ii), and our volume assumption. □

From this, we immediately obtain the result:

Proposition 2.13. We have $e_K \star \mathcal{H}(G) \star e_K = \mathcal{H}(G//K)$

Corollary 2.14. $(\mathcal{H}(G//K), \star)$ *is an algebra*

We proceed with some more basic comments on the structure of these algebras, which show them to be nicer than might be expected.

Lemma 2.15. *For any $g \in G$ KgK is open and compact.*

Proof. Let $J := K \cap g^{-1}Kg$. Then for each $i \in K/J$, choosing $x_i \in K$ such that $x_i J = i$, we have

$$\begin{aligned} KgK &= \bigcup_{k \in K} kgK \\ &= \bigcup_{i \in K/J} x_i gK \end{aligned}$$

Now clearly, for $y \in G$, $x \mapsto yx$ is a homeomorphism, so that each of these left cosets are open and compact. So then also, the left cosets of J in K are open and cover K. So compactness of K dictates that there must be finitely many left cosets. Now then, KgK is a finite union of compact (resp. open) sets, and so compact (resp. open). $\square$

Lemma 2.16. *Any element $f \in \mathcal{H}(G//K)$ is supported on a finite collection of double cosets of K.*

Proof. This is a simple consequence of compact support. $\square$

These results make it obvious that the characteristic functions of double cosets will be incredibly important to our studies.

Notation 2.17. For $g \in G$, when K is clear, we write

$$f_g = \mathbb{1}_{KgK}$$

Now from our previous results, we learn

Proposition 2.18 (Structure of the Hecke Algebra)**.** For G, K as above, the Hecke algebra $\mathcal{H}(G//K)$ is spanned by the elements f_g for $g \in G$.

Proof. From our prior results, it only remains to show that f_g is continuous, but as KgK is open, this is obvious $\square$

2.3 The Representation of Hecke Algebras

Now that we understand the basics of these algebras, we can finalise our definitions so as to prove Theorem 2.1. Before we do this, as we have now defined our algebras, we will take a moment to explain the importance of the Theorem:

Aside (on the importance of Theorem 2.1)**.** Take some irreducible representation $(\pi, V) \in \mathfrak{R}(\mathrm{GL}_n(\mathbb{Q}_p))$. Suppose we have a filtration of the identity in $\mathrm{GL}_n(\mathbb{Q}_p)$ by Moy-Prasad groups (which are indeed open and compact), with nil intersection[3]. As π is smooth, the stabilizer of $v \in V$ is open around the identity and so contains some element of our filtration, K. Then, V^K is non empty, and so as V is irreducible, it is generated by its K-fixed vectors. Then we know by the Theorem that this corresponds to a module of $\mathcal{H}(G//K)$ from which the data of V can be recovered. So, all smooth irreducible representations of $\mathrm{GL}_n(\mathbb{Q}_p)$ are encoded in the representation theory of our Hecke algebras.

[3]These exist, which indeed is the object of the Moy-Prasad filtration.

Of course to prove our theorem, we need to understand the $\mathcal{H}(G//K)$ structure on V^K:

Definition 2.19. Let (π, V) be a smooth representation of G, and K a compact open subgroup of G. Then we write (abusing notation) $\pi : \mathcal{H}(G//K) \to \mathrm{End}(V^K)$,

$$\pi(f) = \int_G f(x)\pi(x)dx$$

In fact, V becomes a $\mathcal{H}(G)$-module by the same method. At this point, alarm bells may be ringing. How are we possibly performing integrals in a possibly infinite dimensional vector space?

Rest assured, these are in fact each finite sums. This is verified by the following series of calculations:

Lemma 2.20. *$f \in \mathcal{H}(G)$ is constant on left cosets of some compact open subgroup K_0*

Proof. Left to the reader[4]. □

Lemma 2.21. *For (π, V) as above and $v \in V$, we have that $v \in V^{K_1}$ for some compact open subgroup K_1 of G*

Proof. This is the combination of smoothness, and that G is *t.d.*. □

Then, $\pi(f)(v) = \mathrm{Vol}(K_0 \cap K_1) \sum_{i=1}^n f(x_i)\pi(x_i)$ for $\mathrm{supp}(f) = \bigcup x_i(K_0 \cap K_1)$, where we have finitely many terms by compactness. Knowing now that we are on firm ground, we can proceed.

Proposition 2.22. The map π gives V^K the structure of a $\mathcal{H}(G//K)$-module.

Proof. The map π is clearly bilinear, so we only need to show that it respects multiplication in the Hecke algebra. This uses essentially the same tricks as our other computations. □

Proposition 2.23. We have $e_K \star V = V^K$

Proof. It is easy to check that $e_K \star V \subset V^K$, and further that e_K fixes V^K point wise. This gives us inclusions both ways. □

[4] *Hint:* Choose a filtration by open compact subgroups of the identity as G is *t.d.*. Use local compactness to construct an open cover of the support on which f is constant, then use compactness.

Definition 2.24. We call a subgroup $K \subset \mathrm{GL}_n(\mathbb{Q}_p)$ a *Moy-Prasad* subgroup when it is part of some Moy-Prasad Filtration of $\mathrm{GL}_n(\mathbb{Q}_p)$, with parameter $r > 0$.

Remark 2.25. A Moy-Prasad Filtration is a filtration of a reductive p-adic group via compact open subgroups, arising from certain data corresponding to the root system. Indeed, we have to specify our theorem to the case of $\mathrm{GL}_n(\mathbb{Q}_p)$ to avoid the definition of a p-adic group. Interested readers will find a gentle introduction in [Fin20], where in particular the Moy-Prasad Filtration is dealt with in Section 3.

Example 2.26. Each set K_i for $i \geq 1$ from Chapter 1 is Moy-Prasad. K_0 however is not.

Remark 2.27. For the theorem above to be true, the element e_K must be what Bushnell-Kutzko ([BK98]) call a *Special Idempotent*. Showing this is true for the groups we are interested in is a frustrating task. The following Proposition of [BS17], shows that all Moy-Prasad subgroups fulfil this criterion.

Proposition 2.28. Suppose $K \subset \mathrm{GL}_n(\mathbb{Q}_p)$ is a Moy-Prasad subgroup, and (π, V) is a representation with $V^K \neq 0$ and V generated by V^K. If V' is a non-trivial subquotient of V, then $V'^K \neq 0$.

Proof. A proof can be found as Proposition 5.1 in the reference above. □

Proof of Theorem 2.1. Throughout the proof we abbreviate $G := \mathrm{GL}_n(\mathbb{Q}_p)$ and $\mathcal{H} := \mathcal{H}(\mathrm{GL}_n(\mathbb{Q}_p))$. To refresh, the definition of our functor M is:

$$M(V) = V^K = e_K \star V$$

To proceed, we first define the inverse functor:

$$N : \mathcal{H}(G//K)\text{-mod} \to \mathfrak{R}_K(G)$$

$$W \mapsto \mathcal{H} \star e_K \otimes_{\mathcal{H}(G//K)} W$$

Where we turn the image into a G-representation via left translation on $\mathcal{H}$. Now notably, from our earlier conversation, we know functions in $\mathcal{H} \star e_K$ are supported on the left cosets of K. So then as $g \cdot (e_K \otimes v) = \mathbb{1}_{gK} \otimes v$, the image is in fact in $\mathfrak{R}_K(G)$. Now, via Proposition 2.13

$$\begin{aligned} W &\cong \mathcal{H}(G//K) \otimes_{\mathcal{H}(G//K)} W \\ &\cong e_K \star \mathcal{H} \star e_K \otimes_{\mathcal{H}(G//K)} W = MN(W) \end{aligned}$$

So it remains to show that $NM(V) \cong V$. We have an obvious map

$$\phi : \mathcal{H} \star e_K \otimes_{\mathcal{H}(G//K)} V^K \to V$$
$$f \otimes v \mapsto \pi(f)v$$

Which is surjective by definition of our category, and as $\pi(\mathbb{1}_{gK})v = gv$. Our only obstruction now is injectivity. Crucially, we have

(i) $\mathrm{Ker}(\phi)$ is a G-subrepresentation of $\mathcal{H} \star e_K \otimes V^K$

(ii) ϕ is injective on $e_K \otimes V^K$

So now

$$\begin{aligned} e_K \star \mathrm{Ker}(\phi) &\subset \mathrm{Ker}(\phi) \cap e_K \star (\mathcal{H} \star e_K \otimes_{\mathcal{H}(G//K)} V^K) \\ &= \mathrm{Ker}(\phi) \cap (e_K \otimes V^K) = 0 \end{aligned}$$

But then, by Proposition 2.28, $\mathrm{Ker}(\phi) = 0$. Naturality is left as an exercise. □

Remark 2.29. Bushnell and Kutzko indeed show that M is equivalence of categories if and only if $\mathfrak{R}_K(G)$ is closed under subquotients.

Remark 2.30. Indeed, A. Borel demonstrates that our theorem is true for the Iwahori subgroup in [Bor76], which we begin to study in the next Chapter. This result is known as the Borel-Casselman equivalence.

2.4 A Lemma on Structure

In our studies, much of the structure of our algebras will be carried over from the structure of specially chosen subgroups. To do this, we extract a useful lemma from [How85]. In preparation, we give a technical definition.

Definition 2.31. We call a locally compact group G *unimodular* when the groups left Haar measure is right invariant

We note that $\mathrm{GL}_n(\mathbb{Q}_p)$ is indeed unimodular, so that the result to follow applies in our case of interest. This is an easy but long exercise to check.

Proposition 2.32. Suppose G is a unimodular group, with $H \subset G$ an open compact subgroup, whose volume is normalized to 1. Suppose $g_1, g_2 \in G$ have

$$\mathrm{Vol}(Hg_1H)\mathrm{Vol}(Hg_2H) = \mathrm{Vol}(Hg_1g_2H)$$

Then using our notation from before, we have

$$f_{g_1} \star f_{g_2} = f_{g_1 g_2}$$

as elements of the Hecke algebra.

The proof is laid out very clearly in [How85] (as Proposition 2.2 in Chapter 3) and so there is little use in replicating it here.

2.5 Example: Spherical Hecke Algebras

Having spent so much reasoning abstractly, we can perform our first calculation of a Hecke algebra. We warn the reader that the groups considered in this Section are not Moy-Prasad.

Example 2.33 ($\mathcal{H}(\mathbb{Q}_p^\times // \mathbb{Z}_p^\times)$)**.** We can fairly easily calculate the disjoint union:

$$\mathbb{Q}_p^\times = \bigcup_{n \in \mathbb{Z}} p^n \mathbb{Z}_p^\times$$

which tells us that f_{p^n} span. At this point, the enthusiastic reader is encouraged to brute force the product structure (which is not very difficult). However normality and our last Proposition tell us that

$$f_{p^i} \star f_{p^j} = f_{p^{i+j}}$$

So clearly $f_p \mapsto X$ defines an isomorphism

$$\mathcal{H}(\mathbb{Q}_p^\times // \mathbb{Z}_p^\times) \cong \mathbb{C}[X, X^{-1}]$$

Which completes our calculation.

Recall briefly

$$K_0 = \mathrm{GL}_n(\mathbb{Z}_p)$$

Our last example, stunningly, generalises to higher dimensions. This is a fact which we opt not to prove here:

Fact 2.34. $\mathcal{H}(G//K_0) \cong \mathbb{C}[X_1, \ldots, X_n, X_n^{-1}]$

This is proved very clearly in the notes [DZ17]. We will be able to say something interesting about simple modules.

Proposition 2.35. We have

$$\mathrm{GL}_n(\mathbb{Q}_p) = \bigcup_{d \in D_-} K_0 d K_0$$

with the monoid D_- defined:

$$D_- = \{d \in \mathrm{GL}_n(\mathbb{Q}_p) : d = \mathrm{Diag}(p^{a_0}, \dots, p^{a_n}), a_0 \geq \dots \geq a_n \in \mathbb{Z}\}$$

Proof. We will show later (as Proposition 4.1) that this is a consequence of the Iwahori Decomposition of $\mathrm{GL}_n(\mathbb{Q}_p)$. □

The reader is advised to swallow anything they might be drinking before reading this next proof, in case they spit it out in delight.

Proposition 2.36. $\mathcal{H}(G//K_0)$ is commutative

Proof. Consider the anti-involution $\iota : \mathcal{H}(G//K_0) \to \mathcal{H}(G//K_0)$ defined via

$$\iota(f)(X) = f(X^T)$$

Obviously, this sends $f_X \mapsto f_{X^T}$. Of course for $d \in D_-$, $\iota(f_d) = f_d$. So as these elements span, ι acts as the identity. But then

$$\begin{aligned} f_d \star f_{d'} &= \iota(f_d \star f_{d'}) \\ &= \iota(f_{d'}) \star \iota(f_d) \\ &= f_{d'} \star f_d \end{aligned}$$

□

From which we learn the following stunning fact.

Proposition 2.37. Suppose (π, V) is an irreducible, admissible representation of $\mathrm{GL}_n(\mathbb{Q}_p)$ admitting a K_0 fixed vector. Then

$$\mathrm{Dim}(V^{K_0}) = 1$$

In future, we will have to work much harder to get similar results for other algebras. We note, as K_0 is not a special idempotent, we used the result of [Bum10] as remarked under Theorem 2.1.

Chapter 3

Affine Reflection Groups

Our primary method of understanding our Hecke algebras in the proceeding notes will be to pull information about the group through to the Hecke algebra by way of Proposition 2.32. In the next Chapter, we will begin to understand the structure of the Iwahori Hecke algebra of $\mathrm{GL}_n(\mathbb{Q}_p)$. We will soon see, the double cosets associated to this Hecke algebra are parametrised by an affine reflection group, with an extra element adjoined. Because of this, to point towards the general method we need at least a rudimentary understanding of affine reflection groups.

3.1 Definition

A reflection group is a finite subgroup of the Orthogonal group of some finite dimensional vector space V, endowed with a positive definite inner product $(\cdot, \cdot)$. Readers interested in reflection groups are directed to [Hum90]. We recall for $v \in V$, a reflection through v fixes the orthogonal hyperplane H_v, while sending v to $-v$. We write the reflection through v as $s_v \in O(V)$.

Definition 3.1. A *root system* $\Phi \subset V$ satisfies

(i) $\Phi \cap \mathrm{Sp}(v) = \{v, -v\}$

(ii) $s_\alpha \Phi = \Phi$ for all $\alpha \in \Phi$

Example 3.2. Consider S_n acting on $\mathbb{R}^n = \mathbb{R}\{\epsilon_i : i \in \{1, \dots, n\}\}$ by permuting the axes. We know S_n is generated by transpositions $(i\ j)$, and indeed by the set $(i\ i+1)$ with $i \in \{1, \dots, n-1\}$. Then the set

$$\Phi = \{\epsilon_i - \epsilon_j : i, j \in \{1, \dots, n\},\ i \neq j\}$$

is a root system, and the group generated by s_α for $\alpha \in \Phi$ is S_n.

Definition 3.3. For a root system $\Phi \subset V$, a *simple system* $\Delta \subset \Phi$ has:

(i) Δ is a basis of V

(ii) $-\mathbb{N}\Delta \cup \mathbb{N}\Delta = \Phi$

Example (Example 3.2 cont.)**.** In the example above,

$$\Delta = \{e_i := \epsilon_i - \epsilon_{i+1} : i \in \{1, \ldots, n-1\}\}$$

is a simple system

From here on, we fix a root system Φ. We define new maps t_α, for $\alpha \in \Phi$ with

$$t_\alpha(x) = x + \frac{2\alpha}{(\alpha, \alpha)}$$

Now we are ready to define the affine reflection group associated to a root system.

Definition 3.4. The *affine reflection group* associated to Φ is the subset of $\mathrm{Aff}(V)$ generated by s_α and t_α for all $\alpha \in \Phi$. Write the affine reflection group associated to the fixed Φ to be $\tilde{W}$.

Notation 3.5. From here on, the affine reflection group associated to Φ in Example 3.2 will be denoted $\tilde{S}_n$.

Now, from the theory of Reflection Groups (see [Hum90] Chapter 4 Section 3), there exists a unique element $\beta \in \mathbb{N}\Delta$ such that for all $\alpha \in \mathbb{N}$, $\beta - \alpha$ is a sum of simple roots. We define $\tilde{s}_\beta := t_\beta \cdot s_\beta$. We quote an important theorem from [Hum90]:

Theorem 3.6. *$\tilde{W}$ is generated by s_α for $\alpha \in \Delta$, and $\tilde{s}_\beta$*

Proof. See [Hum90], Proposition 4.3 of Chapter 4. □

From now on, we call this generating set S.

Example (Example 3.2 cont.)**.** In our case, we write $s_i := s_{e_i}$ and $s_n = \tilde{s}_\beta$, which satisfy for $n > 2$

$$s_i s_j = s_j s_i \text{ for } |i-j| > 1$$

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$$

$$s_i^2 = 1$$

with indices modulo n. When $n = 2$ we omit the first two relations.

Remark 3.7. As stated in the introduction, our aim is to give a minimal proof for the structure of the Iwahori Hecke algebra. In light of this, we have left out two key parts - the Weyl Group, and the Extended Affine Weyl Group. [IM65] generalises these results to Chevalley Groups over a p-adic field, by considering these. Most of the theory that follows is the same. For those familiar with the Weyl Group, it should not be difficult to generalise what we have done.

3.2 The Length Function

Proposition 2.32 is deeply concerned with the volume of double cosets; moreover, we will soon very soon become interested in the Iwahori subgroup I (which we will define shortly). As teased, it turns out in $\mathrm{GL}_n(\mathbb{Q}_p)$ we can parametrize the double cosets of I by the group $\tilde{S_n}$, with an added element. This can all be done in more generality, as is done in [IM65]. To understand the volume of these cosets, we need the following.

Definition 3.8. We define a length function $l : \tilde{W} \to \mathbb{N}_0$ so $l(w)$ is minimal such that w can be written as a product of $l(w)$ elements of S.

Definition 3.9. Define

$$A_0 = \{\lambda \in V : 0 < (\alpha, \lambda) < 1 \text{ for all } \alpha \in \Delta\}$$

Definition 3.10. For $w \in \tilde{W}$, write

$$\mathcal{L}(w) = \{H_s : H_s \text{ seperates } A_0 \text{ and } wA_0\}$$

Now, we state a result from [Hum90]

Proposition 3.11. For $s \in S$ and $w \in \tilde{W}$, $l(sw) > l(w)$ when $H_s \notin \mathcal{L}(W)$

Now specialising to the case of $\tilde{S_n}$.

Proposition 3.12. For $s \in S$ and $w \in \tilde{W}$, $l(sw) > l(w)$ when $(wA_0, e_i) > 0$

3.3 A Crucial Proposition

In this Section, we prove a result of Iwahori that while technical, is the backbone of the rest of our study. From now on, we write $\tilde{W} := \tilde{S_n}$. We embed the affine reflection group $\tilde{W}$ into $\mathrm{GL}_n(\mathbb{Q}_p)$ in the following way.

$$s_i \mapsto \begin{pmatrix} \mathrm{Id}_{i-1} & \cdots & 0 \\ \vdots & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & \vdots \\ 0 & \cdots & \mathrm{Id}_{n-i-1} \end{pmatrix} \text{ for } i \in \{0, ..., n-1\}$$

$$s_n \mapsto \begin{pmatrix} 0 & \dots & p^{-1} \\ \vdots & \mathrm{Id}_{n-2} & \vdots \\ p & \cdots & 0 \end{pmatrix}$$

For later, we introduce an extra element $t \in \mathrm{GL}_n(\mathbb{Q}_p)$

$$t := \begin{pmatrix} 0 & \mathrm{Id}_{n-1} \\ p & 0 \end{pmatrix}$$

Now, $ts_it^{-1} = s_{i+1}$ where i is taken modulo n. So then, as our relations for the Weyl Group obviously hold in the first $n-1$ generators, we can transport them to the "affine" generator using t. So this map genuinely defines an injective group homomorphism. From now on, we tacitly conflate $\tilde{W}$ with its image in $\mathrm{GL}_n(\mathbb{Q}_p)$. It is worth noting that

$$t(x_1, \dots, x_n) \mapsto \mathrm{Diag}(p^{x_1}, \dots, p^{x_n})$$

Now, we finally introduce the object we will soon focus on:

Definition 3.13. Let $U \in \mathrm{GL}_n(\mathbb{F}_p)$ be the subgroup of upper triangular matrices. Then define the *Iwahori Subgroup* of $\mathrm{GL}_n(\mathbb{Q}_p)$

$$I := q_1^{-1}(U)$$

Which we might make explicit via

$$I = \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p & \cdots & \mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p^\times & \ddots & \mathbb{Z}_p & \mathbb{Z}_p \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ p\mathbb{Z}_p & p\mathbb{Z}_p & \ddots & \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p\mathbb{Z}_p & p\mathbb{Z}_p & \cdots & p\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix}$$

Which is open and compact.

We require one calculation before we move on to our crucial Proposition, which we take from [IM65]. This proof is not omitted due first to its importance, and second as work has been done in divorcing the Proposition from the results of the beginning of their paper.

Definition 3.14. For $i \neq j$ and $r \in \mathbb{Z}_p$, we write

$$\mathfrak{X}_{i,j}(r) = \mathrm{Id}_n + rE_{i,j}$$

Lemma 3.15 (Corollary 2.7 of [IM65])**.**

(i) For each $i \in 1, ..., n-1$

$$Is_iI = \bigcup_{r=0}^{p-1} Is_i\mathfrak{X}_{i+1,i}(r)$$

(ii) For s_n *we have*

$$Is_nI = \bigcup_{r=0}^{p-1} Is_nt\mathfrak{X}_{n,n-1}(r)t^{-1}$$

Proof. This is a simple calculation for $i \neq n$ which can be done considering only 2×2 matrix blocks. When $i = n$, we simply transport by t, as it normalizes I. □

Proposition 3.16 (Proposition 2.8 of [IM65])**.** Let $s = s_i$ for some $i \in \{1, \ldots, n\}$ and $w \in \tilde{W}$. Then

(i) If $l(sw) > l(w)$, we have $IsIwI = IswI$,

(ii) If $l(sw) < l(w)$, we have $IsIwI \subset IsI \cup IswI$

Proof. We will prove these for $i < n$, and the case where $i = n$ is similar. We begin with (i). Well, first write $w = t(\lambda)\sigma$, where $\sigma \in S_n$. Then

$$\begin{aligned} IsIwI &= \bigcup_{r=0}^{p-1} Is\mathfrak{X}_{i,i+1}(r)wI \\ &= \bigcup_{r=0}^{p-1} Isw\left(w^{-1}\mathfrak{X}_{i,i+1}(r)w\right) I \end{aligned}$$

So we seek to show $w^{-1}\mathfrak{X}_{i,i+1}(r)w \in I$. But

$$w^{-1}\mathfrak{X}_{i,i+1}(r)w = \mathfrak{X}_{\sigma(i),\sigma(i+1)}(r * p^{(\lambda,e_i)})$$

Where our inner product is an immediate consequence of the embedding of translations into $\mathrm{GL}_n(\mathbb{Q}_p)$. So now, motivated geometrically by Proposition 3.12, we have two possibilities, being that H_s is not separated by A_0 and wA_0:

(i) $\sigma(i) < \sigma(i+1)$ (i.e $\sigma(e_i) \in \mathbb{N}\Delta$) and $(\lambda, e_i) > 0$

(ii) $\sigma(i) > \sigma(i+1)$ (i.e $\sigma(e_i) \in -\mathbb{N}\Delta$) and $(\lambda, e_i) \geq 1$

But in the both cases, our result lands in I.

We proceed to (ii). Suppose that $w = s\sigma$. Then

$$IsIwI = \bigcup_{r=0}^{p-1} Is\mathfrak{X}_{i,i+1}(r)s\sigma I$$

So we investigate $s\mathfrak{X}_{i,i+1}(r)s$. When $r = 0$ this is $I\sigma I$. When $r \neq 0$, we see

$$\begin{array}{ccccccc} \begin{pmatrix}1&1\\0&t\end{pmatrix} & \times & \begin{pmatrix}0&1\\1&0\end{pmatrix} & \times & \begin{pmatrix}1&t^{-1}\\0&-t^{-1}\end{pmatrix} & = & \begin{pmatrix}1&0\\t&1\end{pmatrix} \\ \rotatebox{90}{$\in$} & & \| & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ I & & s & & I & & IsI \end{array}$$

So that, as we may consider this only in the 2×2 blocks, we see that $s\mathfrak{X}_{i,i+1}(r)s \in IsI$ for $r \neq 0$. So then for $r \neq 0$

$$Is\mathfrak{X}_{i,i+1}(r)s\sigma I \subset IsI\sigma I = IwI$$

So as $IsIwI$ is clearly a union of the double cosets that we claimed, we are done. □

Chapter 4

The Iwahori Hecke Algebra

Now we turn our focus to the Iwahori Hecke algebra, which is the Hecke algebra $\mathcal{H}(\mathrm{GL}_n(\mathbb{Q}_p)//I)$ of the Iwahori subgroup I. In this Section, we prove an important theorem on its structure - it is in fact, an *Affine* Hecke algebra. This is proven in [IM65], whose example we follow initially. This allows us to appeal to the rich study of Affine Hecke algebras, about which a great deal is known. From now on, we leave the setting of *t.d.* groups, and finally settle that $G := \mathrm{GL}_n(\mathbb{Q}_p)$.

Remark 4.1. As we know from our earlier work, modules of the Iwahori Hecke algebra correspond to representations of $\mathrm{GL}_n(\mathbb{Q}_p)$ with vectors fixed by I. In fact, every irreducible admissible representations of $\mathrm{GL}_n(\mathbb{Q}_p)$ admitting an Iwahori fixed vector is equivalent to a subquotient of the *spherical principal series* (see the introduction of [Kim99]). These are a special class of representations induced from the upper triangular matrices in $\mathrm{GL}_n(\mathbb{Q}_p)$.

4.1 Iwahori-Bruhat Decomposition

We go on to prove a fascinating theorem of Iwahori and Matsumoto. To do this, we first define

$$W^a = \langle \tilde{W}, t \rangle$$

We are now able to state Theorem 2.16 of [IM65]:

Theorem 4.2 (Iwahori-Bruhat Decomposition)**.** *We have a decomposition*

$$GL_n(\mathbb{Q}_p) = IW^aI = \bigcup_{w \in W^a} IwI$$

where our union is disjoint

Remark 4.3. The name of this might be illuminated by the Bruhat decomposition of, for instance, $\mathrm{GL}_n(\mathbb{F}_p)$:

$$\mathrm{GL}_n(\mathbb{F}_p) = BS_nB$$

Where B is the subgroup of upper triangular matrices. The above can obviously be seen as an extension of this. This is essentially row and column reduction with some extra reordering.

Our strategy will be to show that IW^aI is a subgroup, containing a generating set. The proofs of this Sections take their essence from [IM65].

Lemma 4.4 (Lemma 2.14 of [IM65])**.** *IW^aI is a subgroup*

Proof. It is first crucial to note that t normalises I. So, taking elements $\sigma = t^M s_{c_1} s_{c_2} \dots s_{c_m}, \tau \in W^a$, and assuming our first expression is minimal, we see

$$I\sigma I \times I\tau I = I\sigma I\tau I = t^M I s_{c_1} I \dots I s_{c_m} I\tau I$$

So that we only need to verify $Is_iI\tau I \in IW^aI$ for each i, and $\tau \in W^a$. But Proposition 3.16 above verifies this. □

Lemma 4.5 (Proposition 2.13 of [IM65])**.** *For $\sigma, \tau \in W^a$, $I\sigma I$ and $I\tau I$ are disjoint.*

Proof. Suppose $I\tau I = I\sigma I$, with $l(\tau) \leq l(\sigma)$. We induct on $l(\tau)$.
If $l(\tau) = 0$ then $\tau = t^r$ and $I\sigma I = I\tau$, as t normalizes I. So then $\sigma\tau^{-1} \in I \cap W^a$. It is not hard to see that $I \cap W^a = \{\mathrm{Id}\}$. So we learn $\tau = \sigma$.

Suppose the claim is true for $l(\tau) < n$. Well, we have some s_i such that $l(s_i\tau) < l(\tau)$, and so by Proposition 3.16

$$Is_i\tau I \subset Is_iI\sigma I \subset I\sigma I \cup Is_i\sigma I$$

So then $Is_i\tau I$ is either $I\sigma I$ or $Is_i\sigma I$. But then by induction, either $s_i\tau = \sigma$, or $s_i\tau = s_i\sigma$. But if $s_i\tau = \sigma$, then $l(\sigma) < l(\tau)$. So then clearly $\tau = \sigma$. □

Proof of Theorem 4.2. It is easy to see that G is generated by the set $\{\mathfrak{X}_{i,j}(r)\}$, the diagonal matrices D and the permutation matrices - this is of course because elementary column operations can reduce any invertible matrix to the identity.

We know $(1\ 2\ 3\ \dots\ n) \in \tilde{W} \subset G$ (again conflating $\tilde{W}$ and its image) and can check:

$$\mathrm{diag}(1, 1, \dots, 1, p) = (1\ 2\ 3\ \dots\ n)t \in W^a$$

But then clearly, all diagonal matrices are generated by conjugation of the matrix above by $\tilde{W}$ and I. So IW^aI contains the diagonal matrices.
Finally, we observe that I, D and $\tilde{W}$ generate the set $\{\mathfrak{X}_{i,j}(r)\}$. Obviously, if $i < j$ then $\mathfrak{X}_{i,j}(1) \in I$. So then, we may conjugate by a transposition of $\tilde{W}$ to obtain $\mathfrak{X}_{j,i}(1)$. Moreover, $\mathfrak{X}_{i,j}(r)$ is a conjugation of $\mathfrak{X}_{i,j}(1)$ by a diagonal matrix, so our claim is true. □

So we learn that $\mathcal{H}_I$ is spanned by $\{f_w : w \in W^a\}$. Our next problem will be resolving their multiplication. These cosets have yet more marvellous connections to the group $\tilde{W}$. Before we move on, we will quickly prove an outstanding Proposition of Chapter 2:

Proof. [Proof of 2.35] Take $w \in W^a$, which we may write $w = t(\lambda_1, \ldots, \lambda_n)\sigma$, with σ a permutation matrix. Notably, there is a permutation matrix τ so that $\mathrm{Ad}_\tau t(\lambda) \in D_-$. So then

$$\begin{aligned} IwI &= It(\lambda)\sigma I \\ &= (I\tau^{-1})\mathrm{Ad}_\tau t(\lambda)(\tau\sigma I) \\ &\subset K_0 \mathrm{Ad}_\tau t(\lambda) K_0 \end{aligned}$$

So that clearly $\mathrm{GL}_n(\mathbb{Q}_p) = IW^aI \subset K_0 D_- K_0$. But of course this proves our Proposition. □

4.2 The Volume Property

This Section is dedicated to proving a lovely, and crucial result, in a new way to the author. This result allows us to pull the structure of the group $\tilde{W}$ through to the Hecke algebra. The result is as follows.

Proposition 4.6. For $w \in \tilde{W}$, normalising the Haar Measure so that $\mathrm{Vol}(I) = 1$,

$$\mathrm{Vol}(IwI) = p^{l(w)}$$

Remark 4.7. The Iwahori Decomposition tells us that the Iwahori Subgroup “knows about” W^a - which is not too surprising. This tells us that somehow, the Iwahori Subgroup knows our length function precisely, and so our choice of positive system. Of course, the Iwahori Subgroup “learned about” our choice of positive system when we decided what “Upper Triangular” meant.

Remark 4.8. The above Propositions extends trivially to W^a by the fact that t normalizes I. Discussion of "Extended Weyl Groups" would yield a beautiful generalisation of this fact, but this would not be minimal. This can be found in (1.5) of [Lus89].

To proceed, we first note the following equalities:

$$\mathrm{Vol}(IwI) = \mathrm{Card}(\{Iwy : y \in I\}) = [I : I \cap wIw^{-1}]$$

Proposition 4.9. For $\sigma \in \tilde{W}$ we have $\mathrm{Vol}(I\sigma I) \leq p^{l(\sigma)}$.

Proof. For each $i \in \{0, \ldots, n\}$ we have $\mathrm{Vol}(Is_iI) = p$. Moreover, if $\sigma, \tau \in \tilde{W}$, writing $I\sigma I = \bigcup_{i=1}^{\mathrm{Vol}(I\sigma I)} x_i\sigma I$, we see

$$I\sigma\tau I \subset I\sigma I\tau I = \bigcup_{i=1}^{\mathrm{Vol}(I\sigma I)} x_i I\tau I$$

So that by translation invariance we obtain the following inequality

$$\mathrm{Vol}(I\sigma\tau I) \leq \mathrm{Vol}(I\sigma I)\mathrm{Vol}(I\tau I)$$

and combining these two facts, we learn $\mathrm{Vol}(I\sigma I) \leq p^{l(\sigma)}$. □

We gained one half of our result with relative ease. To get the reverse inequality, we have to get slightly more technical. We include the full details of this next proof, as it is by the author.

Lemma 4.10. *For $w \in \tilde{W}$ we have*

$$Vol(IwI) = p^N$$

for some $N \in \mathbb{N}$

Proof. For this, we reintroduce $q_m : \mathrm{GL}_n(\mathbb{Z}_p) \to \mathrm{GL}_n(\mathbb{Z}/p^n)$, and define

$$A_m := q_m(I \cap wIw^{-1}) \text{ and } B_m := q_m^{-1}(A_m)$$

Claim 1: $[I : B_m] = [q_m(I) : A_m]$.
Proof of 1: Expressing $I = \bigcup_{i=1}^{n} x_i(B_m)$, a disjoint union, we can write

$$\begin{aligned} q_m(I) &= \bigcup_{i=1}^{n} q_m(x_i)\ q_m(B_m) \\ &= \bigcup_{i=1}^{n} q_m(x_i)A_m \end{aligned}$$

where our second equality is by surjectivity of q_m. Now if $q(x_i)A_m = q(x_j)A_m$ then $q(x_i x_j^{-1}) \in A_m$. So then $x_i x_j^{-1} \in B_m$, but this is absurd. $\square$−1

Claim 2: $[q_m(I) : A_m]$ is a power of p.
Proof of 2: The diagonal matrices are normalized by $\tilde{W}$, and so diagonals in $\mathrm{GL}_n(\mathbb{Z}/p^m)$ are contained in each A_m. Further, by considering the determinant, the image of I consists of matrices with units down their diagonal, and with no conditions elsewhere. Therefore the index of the diagonal matrices in $\mathrm{GL}_n(\mathbb{Z}/p^m)$ must be a power of p. $\square$−2

Claim 3: $\bigcap_{m\in\mathbb{N}} B_m = I \cap wIw^{-1}$
Proof of 3: For $b \in \bigcap_{m\in\mathbb{N}} B_m$, and each $m \in \mathbb{N}$ there is $b_m \in I \cap wIw^{-1}$ such that $b_m - b \in \ker q_m$. So as $\mathrm{diam}(\ker q_m) \leq n\sqrt{p^{-m}}$, clearly b is a limit point of $I \cap wIw^{-1}$. As $I \cap wIw^{-1}$ is closed, $b \in I \cap wIw^{-1}$. $\square$−3
Now clearly,

$$[I : I \cap wIw^{-1}] = [I : B_m][B_m : I \cap wIw^{-1}]$$

so we see $[B_m : I \cap wIw^{-1}]$ must stabilize to 1 for large m, giving our result. $\square$

Now, we only need one final lemma

Lemma 4.11. *Take $s_i, \sigma \in \tilde{W}$ and suppose that $l(s_i\sigma) > l(\sigma)$. Then*

$$Vol(Is_i\sigma I) > Vol(I\sigma I)$$

Proof. Let $s := s_i$ and suppose for contradiction that Vol$(Is\sigma I)$ $\leq$Vol$(I\sigma I)$. Moreover, suppose that Vol$(I\sigma I) = p^N$. We may then write

$$IsI = \bigcup_{j=1}^{p} x_j sI, \quad I\sigma I = \bigcup_{j=1}^{p^N} y_j \sigma I$$

By Proposition 3.16, $IsI\sigma I = Is\sigma I$, and we may write

$$Is\sigma I = \bigcup_{i,j} x_i s y_j \sigma I$$

Suppose first that Vol$(Is\sigma I)$ <Vol$(I\sigma I)$. The volume of $Is\sigma I$ may be at most p^{N-1}. So then, by the pigeon-hole principle, we must have some $i \in \{1, \ldots, p\}$ and
$j \neq j' \in \{1, ..., p^N\}$ such that

$$x_i s y_j \sigma I = x_i s y_{j'} \sigma I$$

But this is absurd. So we may assume Vol$(Is\sigma I)$ =Vol$(I\sigma I)$.
To avoid the scenario we just encountered, we must have that each left I coset of $Is\sigma I$

contains each x_i exactly once. So then, for each $m \in \{1, ..., p^N\}$ and $i \in \{1, \ldots, p\}$, we have $m_i \in \{1, ..., p^N\}$ such that

$$sy_{m_i}\sigma I = x_i s y_m \sigma I$$

taking $x_1 = \mathrm{Id}$. Now then, taking a union over m

$$sx_i s I \sigma I \subset I \sigma I$$

And further, taking a union over all i,

$$sIsI\sigma I = s(\bigcup x_j s I \sigma I) \subset I \sigma I$$

But now, we once again appeal to Proposition 3.16:

$$I\sigma I \supset IsIsI\sigma I = IsIs\sigma I = Is\sigma I \cup I\sigma I$$

but the Iwahori-Bruhat decomposition contradicts this. □

Proof of Proposition 4.6. Our last two lemmas show that

$$Vol(I\sigma I) \geq p^{l(\sigma)}$$

As our penultimate lemma tells us increases in length only occur by factors of p, and our final lemma tells us the volume does in fact increase as the length increases. □

4.3 Structure of the I-H Algebra

Having proven our rather remarkable result on volumes, it is now fairly simple to compute the structure of our Hecke algebra.

Theorem 4.12 (Structure of the Iwahori Hecke algebra)**.** *The algebra $\mathcal{H}(G//I)$ is generated by f_1, f_t, and f_{s_i}, for $i = 1, \ldots, n$, with relations*

(i) $f_{s_i} \star f_{s_j} = f_{s_j} \star f_{s_i}$ *for* $|i - j| > 1$

(ii) $f_{s_i} \star f_{s_{i+1}} \star f_{s_i} = f_{s_{i+1}} \star f_{s_i} \star f_{s_{i+1}}$

(iii) $f_t \star f_{s_i} \star f_{t^{-1}} = f_{s_{i+1}}$

(iv) $f_{s_i}^2 = (p-1) f_{s_i} + p f_1$

where f_1 acts as the identity.

Remark 4.13. We note as in the continuation of Example 3.2, we omit relations (i) and (ii) at $n = 2$.

Proof. First, Proposition 2.32 tells us that, with reference to Proposition 4.6, any element $w = s_{c_1} \star \ldots \star s_{c_m} t^M \in W^a \backslash \{\text{Id}\}$ with $l(w) = m$ has

$$f_w = f_{s_{c_1}} \star \ldots \star f_{s_{c_m}} \star f_t^M$$

so that these indeed generate the algebra.
We now justify the relations. Our first two relations come from the same Propositions, as these products respect the length function of the group (where again we refer to the continuation of Example 3.2). We then only need to verify our final relation. Dropping the subscript on s_i for convenience,

$$\begin{aligned} f_s \star f_s(x) &= \int_G f_s(y) f_s(y^{-1}x) dy \\ &= \text{Vol}(y : y \in IsI \text{ and } y^{-1}x \in IsI) \end{aligned}$$

Now with such x, y it is clear to see

$$x \in yIsI \subset IsIsI = I \cup IsI$$

so that $f_s \star f_s$ is supported on $I \cup IsI$. We thus only need to find the value it takes at 1 and s. First we easily obtain

$$f_s \star f_s(1) = \int_G f_s(y) f_s(y^{-1}) dy = \text{Vol}(IsI) = p$$

Proceeding with more difficulty

$$\begin{aligned} f_s \star f_s(s) &= \int_G f_s(y) f_s(y^{-1}s) dy \\ &= \text{Vol}(\{y \in G : y \in IsI \text{ and } sy^{-1} \in IsI\} \end{aligned}$$

crucially, for such y, we have $y \in IsI \implies sy^{-1} \in IsIsI$. So, noticing that this only depends on the right coset of I, writing

$$IsI = \cup_{i=1}^{p} x_i s I$$

we only need to check when $sx_i s \in I$. By definition of our x_i, which span $I/(I \cap sIs)$, this is only the case for the one element representing the identity coset. So we have $f_s \star f_s(s) = p - 1$, and we are done. □

4.4 Example: The I-H Algebra at $n = 2$

Though it is not obvious without knowing their definition, we have just shown that the Iwahori Hecke algebra is an Affine Hecke algebra. The theory of these algebras is incredibly rich, and their representation theory is well understood. As an extended "example", we demonstrate the existence of a large commutative subalgebra of the Iwahori Hecke algebra, allowing us to bound the dimension of its irreducible modules. We restrict ourselves to $n = 2$, and follow the path found in [Lus89]. There, what proceeds is done in more (painful) generality for Affine Hecke algebras. Our proofs are different, owing to their specificity.
We will not define precisely what an Affine Hecke algebra is as to do so would require a deeper discussion of root systems, which would take us too far afield. This may be found in [Lus89].

Remark 4.14. Doing this for larger n is not theoretically demanding, but we would soon find ourselves with pages of identities relating to the length function.

Notation 4.15. As suggested, from now on $G := \mathrm{GL}_2(\mathbb{Q}_p)$. We set a precedent for the remainder of our discussion. We denote elements of the group G in round parentheses, so that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$$

In contrast - we denote the characteristic function supported on these double cosets with square parentheses

$$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] := f_{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \in \mathcal{H}(G//I)$$

So that the bulk of our calculation is not performed in subscripts. We believe this notation should be sufficiently distinct so that confusion should not arise.

For now, we work in the Extended Weyl Group W^a, renaming the elements for convenience:

$$s := s_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \;\; r := s_2 = \begin{pmatrix} 0 & p^{-1} \\ p & 0 \end{pmatrix}; \;\; t = \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}$$

Where we have renamed s_1, s_2. For notational consistency with [Lus89] we will write

$$T_x := \begin{pmatrix} p^a & 0 \\ 0 & p^b \end{pmatrix} = t(x)$$

for $x = \begin{pmatrix} a \\ b \end{pmatrix}$, associating diagonal matrices with the translations in $\tilde{W}$. Now, we define

$$\mathbb{Z}^2_{\text{Dom}} = \{x \in \mathbb{Z}^2 : (x, e_1 - e_2) \geq 0\}$$

The subspace of the lattice $\mathbb{Z}^2$ that has non-negative inner product with all of our positive roots (of course, at $n = 2$ we only have one). It is not difficult to see that we may write any element in $\mathbb{Z}^2$ as a difference of two such elements. We have the following fact (which is where our length functions would start to get messy):

Proposition 4.16. For $x, y \in \mathbb{Z}^2_{\text{Dom}}$ we have $[T_x][T_y] = [T_{x+y}]$

Proof. A calculation shows that for $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{Z}^2_{\text{Dom}}$

$$T_{\begin{pmatrix} a \\ b \end{pmatrix}} = t^{2b}(st)^{a-b}$$

And again,

$$(st)^n = \begin{cases} t^n(sr)^{\frac{n}{2}} \text{ for } n \text{ even} \\ t^n r(sr)^{\frac{n-1}{2}} \text{ for } n \text{ odd} \end{cases}$$

The expression we obtain is clearly reduced (as sr, rs have infinite order). It is an easy to check that multiplication of these elements respects the length function (i.e that the lengths add), so that by Proposition 2.32 we get our result. □

So then, it becomes relatively clear why we are interested in this strange subset.

Definition 4.17. For $x \in \mathbb{Z}^2$, write $x = x_1 - x_2$ for $x_1, x_2 \in \mathbb{Z}^2_{\text{Dom}}$. Write

$$\bar{T}_x = [T_{x_1}][T_{x_2}]^{-1}$$

Remark 4.18. It is crucial to note here that the elements of the Affine Hecke algebra are in fact invertible, allowing us to carry out this procedure. We will construct inverses of the generators explicitly soon.

Remark 4.19. These elements are a crucial part of the Bernstein-Lusztig presentation of an Affine Hecke algebra, which describes not only the large commutative sub-algebra demonstrated here, but indeed its centre. These results are found in Section 3 of [Lus89].

We continue with a vital Proposition

Proposition 4.20. For $x, y \in \mathbb{Z}^2$ with $x = x_1 - x_2$ as before:

(i) $\bar{T}_x$ does not depend on choice of x_1, x_2.

(ii) $\bar{T}_x \bar{T}_y = \bar{T}_{x+y}$

Proof.

(i) This is immediate from our previous Proposition.

(ii) Let $y = y_1 - y_2$, with $y_1, y_2 \in \mathbb{Z}^2_{\text{Dom}}$. Then

$$\begin{aligned}\bar{T}_x \bar{T}_y &= [T_{x_1}][T_{x_2}]^{-1}[T_{y_1}][T_{y_2}]^{-1} \\ &= [T_{x_1}][T_{y_1}][T_{y_1}]^{-1}[T_{x_2}]^{-1}[T_{y_1}][T_{y_2}]^{-1} \\ &= T_{x_1}[T_{y_1}][T_{x_2}]^{-1}[T_{y_1}]^{-1}[T_{y_1}][T_{y_2}]^{-1} \\ &= [T_{x_1+y_1}][T_{x_2+y_2}]^{-1} \\ &= \bar{T}_{x+y}\end{aligned}$$

Where the second equality follows as we know $[T_{x_2}]$ and $[T_{y_1}]$ commute.

□

We are ready to define our large subalgebra:

Definition 4.21. We let

$$\mathcal{O} = \mathbb{C}[\bar{T}_x : x \in \mathbb{Z}^2]$$

Clearly from our previous Proposition, this is a commutative subalgebra. Our procedure here has been pretty much complete in terms of generality - only our proofs are easier.

Example 4.22. We have the following identities

$$\bar{T}_{\begin{pmatrix}1\\0\end{pmatrix}} = [s][t]; \quad \bar{T}_{\begin{pmatrix}1\\1\end{pmatrix}} = [t]^2; \quad \bar{T}_{\begin{pmatrix}0\\1\end{pmatrix}} = \frac{1}{p}([r][t] - (p-1)[t])$$

The only non-trivial statement is the third. We note that this is indeed the inverse of $t^{-1}r$, as $[r]^{-1} = \frac{1}{p}([r] + (p-1))$

Proposition 4.23. $\mathcal{H}(G//I)$ is finitely generated as a left $\mathcal{O}$-module.

Proof. We claim that as a left module $\mathcal{H}(G//I) \cong \mathcal{O} \oplus \mathcal{O}t$. We need to show that this contains each element of our basis. This is a routine calculation, where we use

$$([r][t] - (p-1)[t])^{2i} = (rs)^i t^{2i} + \text{terms in } r, s \text{ of order less than 2i}$$

and various other similar formulas, to inductively construct our basis. Verification is left as an exercise to the careful reader. □

A convenient use of the Tensor-Hom adjunction gives us a welcome bound:

Proposition 4.24. Suppose (π, V) is an irreducible, admissible representation of G admitting an I fixed vector. Then

$$\mathrm{Dim}(V^I) \leq 2$$

Proof. For now, we abbreviate $\mathcal{H} := \mathcal{H}(G//I)$. From the remark under Theorem 2.1, using the statement of [Bum10], we know that V^I is a simple $\mathcal{H}$-module (we could have also cited the Borel-Casselman equivalence - see Remark 2.30). Well, the restriction of V^I to an $\mathcal{O}$-module must contain some simple $\mathcal{O}$-module W, which will notably be 1 dimensional. So then we get via the Tensor-Hom Adjunction

$$0 \neq \mathrm{Hom}_{\mathcal{O}}(W, V^I) \cong \mathrm{Hom}_{\mathcal{O}}(W, \mathrm{Hom}_{\mathcal{H}}(\mathcal{H}, V^I)) \cong \mathrm{Hom}_{\mathcal{H}}(\mathcal{H} \otimes_{\mathcal{O}} W, V^I)$$

So there is a map $\mathcal{H} \otimes_{\mathcal{O}} W \to V^I$, but as our image is irreducible, it must surely be a surjection. So

$$\mathrm{Dim}(V^I) \leq \mathrm{Dim}(\mathcal{H} \otimes_{\mathcal{O}} W) = 2$$

From our last Proposition. □

Chapter 5

Congruence Hecke Algebras

Having explored the Iwahori Hecke algebra, in this Chapter we "enlarge" our algebras using a filtration of the Iwahori subgroup. As discussed in Chapter 2, the Hecke algebras of a filtration of the identity by open compact subgroups will necessarily contain the complete data of representations of $\mathrm{GL}_n(\mathbb{Q}_p)$. As such, we have less hope of understanding them as completely as the Iwahori Hecke algebra. In this Section, we present two calculations performed by the author.

5.1 Howe's Presentation at $n = 2$

To begin, we define the filtration of I. We start by assigning

$$\mathcal{I} = q_1^{-1}(U) = \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}$$

where U is the ring of upper triangular matrices in $\mathrm{GL}_2(\mathbb{F}_p)$. Then, define for $l \geq 1$

$$I_m = \mathrm{Id}_2 + p^m\mathcal{I} = \begin{pmatrix} 1 + p^m\mathbb{Z}_p & p^m\mathbb{Z}_p \\ p^{m+1}\mathbb{Z}_p & 1 + p^m\mathbb{Z}_p \end{pmatrix}$$

which has

$$I \supset I_1 \supset I_2 \supset I_3 \supset \dots$$

$$\bigcap_{i \in \mathbb{N}} I_i = \{\mathrm{Id}\}$$

Note further that I_m is normal in I. We will elucidate the structure of I/I_m as it becomes relevant later.

Remark 5.1. For $l \geq 1$ I_m is indeed Moy-Prasad, allowing us to use our Theorem. This is in fact Example 2 of Section 3.1 in [Fin20].

Remark 5.2. In [How85] Howe and Moy gave a presentation of the algebras $\mathcal{H}(G//I_m)$, which we specialize to $n = 2$. These algebras were utilised in their proof that a map between certain Hecke algebras of GL_n over different finite extensions of $\mathbb{Q}_p$ is an isomorphism. Of course, in light of Theorem 2.1, these algebras are of independent interest. The details of this are found in Chapter 3 of [How85].

We recycle our notation from the previous Chapter. Here $[x] = f_{I_m x I_m}$.

Theorem 5.3 (Structure of the Congruence algebras)**.** *$\mathcal{H}(GL_2(\mathbb{Q}_p)//I_m)$ is generated as an algebra by*

(i) $[s]$, $[r]$

(ii) $[t]$, $[t^{-1}]$

(iii) $[x]$ *for* $x \in I$,

subject to relations:

A: $[s]^2 = p\sum_{a=0}^{p-1}\left[\begin{pmatrix}1 & 0\\ ap^m & 0\end{pmatrix}\right]$

B: *(i) For* $a, d \in \mathbb{Z}_p^\times$ *and* $b, c \in \mathbb{Z}_p$

$$[t]\left[\begin{pmatrix}a & b\\ pc & d\end{pmatrix}\right][t^{-1}] = \left[\begin{pmatrix}d & c\\ pb & a\end{pmatrix}\right]$$

(ii) $[t]s[t^{-1}] = r$

(iii) $[t]r[t^{-1}] = s$

C: *(i)* $\left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right]$ *is the Identity of* $\mathcal{H}(G//I_l)$.

(ii) $[x] \star [y] = [xy]$ *for* $x, y \in I$

(iii) For $a, d \in \mathbb{Z}_p^\times$ *and* $b, c \in \mathbb{Z}_p$

(a) $[s]\left[\begin{pmatrix}a & pb\\ pc & d\end{pmatrix}\right] = \left[\begin{pmatrix}d & pc\\ pb & a\end{pmatrix}\right][s]$

(b) $[r]\left[\begin{pmatrix}a & b\\ p^2c & d\end{pmatrix}\right] = \left[\begin{pmatrix}d & c\\ p^2b & a\end{pmatrix}\right][r]$

(iv) For $x \in \mathbb{Z}_p^\times$

$$[s]\left[\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right][s] = p\left[\begin{pmatrix}1 & x^{-1}\\ 0 & 1\end{pmatrix}\right][s]\left[\begin{pmatrix}x & 0\\ 0 & -x^{-1}\end{pmatrix}\right]\left[\begin{pmatrix}1 & x^{-1}\\ 0 & 1\end{pmatrix}\right]$$

Remark 5.4. This theorem is certainly not obvious, and not easy to digest at first sight. Roughly, to the Iwahori Hecke algebra we have "added" the group I/I_m (realised as the elements f_I), and made the squaring of our involutive generators, along with their interaction with I more arcane.

Proof of Theorem 5.3. As we expounded the structure of the Iwahori Hecke algebra in such exhaustive detail, little more is gained by proving this, so it is left to Howe. Howe paves the same path of calculating volumes and supports, as we did prior. This is found in Chapter 3, Section 2. □

5.2 Results

Finally we can move onto the study of these algebras. We begin with some facts that will serve us well throughout our study:

Proposition 5.5. We have the following:

(i) $\mathbb{Z}_p^\times = \{x \in \mathbb{Z}_p : \pi_1(x) \neq 0\}$

(ii) $I = \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix}$

(iii) I_m is normal in I for each $m \in \mathbb{N}$.

Proof.

(i) Because $\mathbb{F}_p$ is a field and π_1 is surjective, we can specialise to the case where $\pi_1(x) = 1$. Take $\pi_1(x) = 1$. Then we may write $x = 1 + px'$, and von-Neumann series will give:

$$\frac{1}{1+px'} = \sum_{i=0}^{\infty} (-1)^i (px)^i$$

So that x is clearly invertible.

(ii) Consider (1), and that our determinant must be a unit.

(iii) We seek to show for $a, d \in \mathbb{Z}_p^\times$ and $b, c \in \mathbb{Z}_p$

$$\begin{pmatrix} a & b \\ pc & d \end{pmatrix} I_m = I_m \begin{pmatrix} a & b \\ pc & d \end{pmatrix}$$

But writing $I_m = 1 + p^m \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}$ we only need show that

$$\begin{pmatrix} a & b \\ pc & d \end{pmatrix} \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix} = \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix} \begin{pmatrix} a & b \\ pc & d \end{pmatrix}$$

Which follows immediately, knowing that a, d are units.

□

5.2.1 1-Dimensional Representations

For notational convenience, we fix $m \in \mathbb{N}$ and denote $\mathcal{H} := \mathcal{H}(G//I_m)$.
First, we construct all 1-dimensional representations when $p \neq 2$. We note that relation C(ii) above tells us that for $x \in I$

$$x \mapsto [x] = f_{I_m x I_m}$$

defines a group homomorphism to it's image. Obviously this descends to a map

$$I_m \backslash I / I_m \cong I/I_m \to \mathcal{H}$$

Where we have used that I_m is normal in I. This is now a group isomorphism onto its image, so that we have

$$\mathbb{C}(I/I_m) \subset \mathcal{H}$$

So certainly a 1-dimensional representation of $\mathcal{H}$ restricts to a 1-dimensional representation of I/I_m.

Proposition 5.6. *I/I_m is isomorphic to the set*

$$M = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}(\mathbb{Z}/p^m) : a, b \not\equiv 0 (\mathrm{mod}\ p) \right\}$$

with multiplication

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} * \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} aa' + pbc' & ab' + bd' \\ ca' + dc' & pcb' + dd' \end{pmatrix}$$

Where we have used $*$ to denote the twisted multiplication.

Proof. We see

$$\begin{aligned} \begin{pmatrix} a & b \\ pc & d \end{pmatrix} I_m &= \begin{pmatrix} a & b \\ pc & d \end{pmatrix} \left(1 + p^m \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix} \right) \\ &= \begin{pmatrix} a & b \\ pc & d \end{pmatrix} + p^m \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix} \end{aligned}$$

So that the map

$$\begin{pmatrix} a & b \\ pc & d \end{pmatrix} I_m \mapsto \begin{pmatrix} a \pmod{p^m} & b \pmod{p^m} \\ c \pmod{p^m} & d \pmod{p^m} \end{pmatrix}$$

is a group isomorphism. The p in the lower left causes our bizarre multiplication. □

Remark 5.7. We will not use this at all past our next Proposition, where we use this embedding to easily verify the cardinalities calculated in the proof.

Remark 5.8. Our next Proposition has the restriction that $p \neq 2$ - this restriction comes from our calculation of the commutator subgroup. Notably, at $p = 2$, $m = 1$, I/I_m is Abelian, and so the calculation we provide below is certainly wrong.

Notation 5.9. We write $F := I/I_m$, with multiplication as dictated above.

Proposition 5.10. *For $p \neq 2$, 1-dimensional representations of F are in correspondence with group homomorphisms $D \longrightarrow \mathbb{C}^\times$ trivial on elements*

$$\begin{pmatrix} 1+py & 0 \\ 0 & (1+py)^{-1} \end{pmatrix} \quad \text{for } y \in \mathbb{Z}/p^n$$

where we write $D \subset F$ for the subgroup of diagonal matrices.

Proof. Recall that 1 dimensional representations of F are in correspondence with representations of $F/[F, F]$. We seek to determine this quotient. We note

$$\left[\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \left[\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

Then, writing

$$K = \left\langle \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle$$

we have that $K \trianglelefteq [F, F]$. Moreover, we have the following crucial equation:

$$\begin{pmatrix} \lambda & \mu \\ \pi & \nu \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \nu - p\lambda^{-1}\mu\pi \end{pmatrix} * \begin{pmatrix} 1 & 0 \\ -(\nu - p\lambda^{-1}\mu\pi)^{-1}\pi & 1 \end{pmatrix} * \begin{pmatrix} 1 & \lambda^{-1}\mu \\ 0 & 1 \end{pmatrix} \tag{5.1}$$

Immediately, this shows the diagonal matrices span the cosets of $[F, F]$. So we have a surjective group homomorphism:

$$D \xrightarrow{\varphi} F/K$$

showing our image is Abelian, and so $[F, F] = K$.
We now need to understand $\text{Ker}(\varphi)$. First note that

$$\begin{pmatrix} 1+py & 0 \\ 0 & (1+py)^{-1} \end{pmatrix} \in K$$

which we find by utilising Equation 5.1 on $\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. This gives us a lower bound on the size of $\text{Ker}(\varphi)$.

Now, observe the following two easily verified facts:

(i) The cosets of K have constant determinant,

(ii) The cosets of K have constant residue modulo p in their $(1, 1)$ entry.

So as the diagonal matrices contain elements attaining each of the $(p^m - p^{m-1})(p-1)$ values, we have at least as many elements of F/K. So then,

$$p^{m-1} \leq |\text{Ker}(\varphi)| = \frac{|D|}{|(F/K)|} \leq \frac{(p^m - p^{m-1})^2}{(p-1)(p^m - p^{m-1})} = p^{m-1},$$

and this is in fact an equality. So then immediately:

$$\text{Ker}(\varphi) = \left\{ \begin{pmatrix} 1+py & 0 \\ 0 & (1+py)^{-1} \end{pmatrix} : y \in \{0, \dots p^{n-1} - 1\} \right\}$$

So 1-dimensional representations of F are in correspondence with characters of $D/\text{Ker}(\varphi)$, which are of the required form. □

Remark 5.11. Notably, representations of F are gained from a group homomorphism $\phi : D \to \mathbb{C}^\times$ by extending to F via Equation 5.1, trivial on K.

We have nearly achieved our goal. We however, have one more restriction. We note, that a representation $\rho : \mathcal{H} \to \mathbb{C}$ must obey

$$\rho\left([t]\left[\begin{pmatrix} a & b \\ pc & d \end{pmatrix}\right][t^{-1}]\right) = \rho\left(\left[\begin{pmatrix} a & b \\ pc & d \end{pmatrix}\right]\right)$$

Where in particular, restricting to diagonal matrices:

$$\rho\left(\left[\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\right]\right) = \rho\left(\left[\begin{pmatrix} d & 0 \\ 0 & a \end{pmatrix}\right]\right)$$

So that the restriction of ρ to $\mathbb{C}(I/I_m)$ obeys this. This subsumes our last criterion. Utilising Equation 5.1 once more, we see

$$\rho\left(\left[\begin{pmatrix} a & b \\ pc & d \end{pmatrix}\right]\right) = \rho\left(\left[\begin{pmatrix} a & 0 \\ 0 & d - pa^{-1}bc \end{pmatrix}\right]\right) = \rho\left(\left[\begin{pmatrix} ad - pbc & 0 \\ 0 & 1 \end{pmatrix}\right]\right)$$

So this restriction must be of them form

$$\rho : \mathbb{C}(I/I_m) \to \mathbb{C}^\times$$

$$[M] \mapsto \phi(\pi_m(\det(M)))$$

For the homomorphism $\phi : (\mathbb{Z}/p^n)^\times \to \mathbb{C}^\times$, where

$$\phi(a) = \rho\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)$$

Proposition 5.12. The 1-dimensional representations of $\mathcal{H}(G//I_m)$ are given by

1. A group homomorphism $\phi : (\mathbb{Z}/p^n)^\times \to \mathbb{C}$.
2. An element $\tilde{t} \in \mathbb{C}$

Where $\rho : \mathcal{H} \to \mathbb{C}$ is given by

$$\rho([s]) = \pm p; \quad \rho([r]) = \pm p; \quad \rho([t]) = \tilde{t}$$

$$\rho([M]) = \phi(\det(M)) \text{ for } M \in I$$

Proof. From the preceding discussion, we only need to establish $\rho([r])$, $\rho([s])$ and $\rho([t])$ (we note $[t]^{-1} = [t^{-1}]$ by *B(i)*). The equations for squaring $[s]$ and $[r]$ clearly fix their image - $[t]$ however may be mapped freely. Now we only need to verify that such maps are representations; that is, they obey the relations set out earlier. This basic computation is left to the reader. □

We note of course that by Theorem 2.1, this immediately constructs all representations of $\mathrm{GL}_n(\mathbb{Q}_p)$ with some 1-dimensional I_m fixed space, for any $m \in \mathbb{N}$.

5.2.2 Bounds and Quotients

In this Section, we seek to improve on bounds given by Bernshtein in [Ber74], for the dimension of irreducible $\mathcal{H}(G//I_m)$-modules. Indeed, Bernshtein proves the following:

Theorem 5.13. *Take K a compact open subgroup of $GL_n(\mathbb{Q}_p)$. Then there exists an integer $N = N(G, K)$ such that all smooth irreducible representations of $GL_n(\mathbb{Q}_p)$ admitting a K-fixed vector have*

$$Dim(V^K) \leq N$$

As we specialise our efforts to I_m, and $n = 2$ we can obtain tighter bounds than Bernshtein. Moreover, we can improve on his bounds further by considering a quotient algebra. Bernshtein's proof relies of the following fact of algebra:

Proposition 5.14. Let $\mathcal{L}$ be a unital $\mathbb{C}$-algebra, with subalgebras $\mathcal{A}, \mathcal{Z} \subset \mathcal{L}$. Take $A_1, ..., A_l \in \mathcal{A}$, $X_1, ..., X_a, Y_1, ..., Y_b \in \mathcal{L}$. Suppose $\mathcal{Z}$ lies in the centre of $\mathcal{L}$ and $\mathcal{A}$ is generated by $A_1, ..., A_l$. Suppose further that any element of $\mathcal{L}$ can be written as $\sum X_a P_{a,b} Y_b$ for $P_{a,b} \in \mathcal{A}$. Then any irreducible finite-dimensional representation of the algebra $\mathcal{L}$ has dimension at most $(ab)^{2^{l-1}}$

We calculate Bernshtein's bound in our case, via his method. As can be seen via their Example (which falls into Case 1.) we obtain parameters relevant to the previous Proposition:

$$l = 1, \quad a = b = [K_0 : I_m].$$

Writing U for the upper triangular matrices in $\mathrm{GL}_n(\mathbb{Q}_p)$ we see

$$[K_0 : I] = [\mathrm{GL}_2(\mathbb{F}_p) : U] = p + 1$$

So that $[K_0 : I_m] = (p+1)(p^m - p^{m-1})^2 p^{2m}$. Thus Bernshtein's bound is

$$N(G, I_m) \leq (p+1)^2(p-1)^4 p^{8m-4}$$

We attempt to refine this, with our finer understanding of the algebra. Careful consideration of our relations (which is done in detail in Chapter 3, Section 2 of [How85]) shows that $\mathcal{H}$ is spanned by elements $[x][w][y]$, for $x, y \in I$ and $w \in \tilde{W}$. We let $\mathcal{Z}$ the algebra spanned by t^2, and $\mathcal{A}$ be the algebra spanned by the element $a_1 := sr$ and $\mathcal{Z}$. Then, letting $X_1, ..., X_a$ be the elements in $\mathcal{H}$ supported on:

$$I/I_m, \ I/I_m[r], \ [t]I/I_m, \ [t]I/I_m[r]$$

and the $Y_1, ..., Y_b$ be the elements supported on:

$$I/I_m, \ [s]I/I_m$$

we get $a \leq 4p^{2m}(p^m - p^{m-1})$ and $b \leq 2p^{2m}(p^m - p^{m-1})$. So then our new refined bound is

$$N(G, I_m) \leq 8(p-1)^4 p^{8m-4}$$

Which is strictly less than Bernshtein's bound (though in some cases, not by very much).

Remark 5.15.

(i) Here, we go to great pains to avoid increasing our parameter l, as our bound loosens exponentially with its growth.

(ii) The inequalities on our parameters a and b stem from the fact that $[s]$ and $[r]$ kill elements of I/I_m (which we will use below).

We seek to improve this bound further, by considering ideals of $\mathcal{H}$. For this, write

$$u(x) = \left[\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right], \ v(x) = \left[\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\right]$$

and define

$$\mathcal{N} := \bigoplus_{a,b\in\{0,\ldots,p-1\}} (1 - v(bp^m))(1 - u(ap^{m-1}))\mathbb{C}(I/I_m)$$

$$\mathcal{M} = \bigoplus_{n\in\mathbb{Z}} \mathcal{N}[t]^n$$

We then have the following

Proposition 5.16. We have the following

(i) $\mathcal{M}$ is a left ideal in $\mathcal{H}(G//I_m)$

(ii) $\mathrm{Dim}_{\mathbb{C}}(\mathcal{N}) < p^{2m}(p^m - p^{m-1})^2$

Proof. We organise our proof of (i) into 2 claims:
Claim 1: $\mathcal{M}$ is invariant under left multiplication by $[x]$ for $x \in I$.
Proof of Claim 1: We already know from Equation 5.1 that any element of $\mathbb{C}(I/I_m)$ may be written as a product of a lower triangular, an upper triangular, and a diagonal matrix. So we only need to check matrices of these forms. We show that for any $M \in I$ for each $a, b \in \{0, \ldots, p-1\}$ there are $a', b' \in \{0, \ldots, p-1\}$ and $M' \in I$ such that

$$[M](1 - v(bp^m))(1 - u(ap^{m-1})) = (1 - v(b'p^m))(1 - u(a'p^{m-1}))[M']$$

Clearly for M diagonal the above holds with

$$M' = M, \ \ a' = (M_{11}M_{22}^{-1})a, \ \ b' = (M_{11}^{-1}M_{22})b.$$

Moreover, all upper (resp. lower) triangular matrices commute with elements $u(a)$ (resp $v(a)$) for $a \in \mathbb{Z}_p$. We now only need to verify that lower (resp. upper) triangular matrices will commute with $u(ap^{m-1})$ (resp $v(ap^m)$). This is a straightforward computation:

$$\left[\begin{pmatrix} 1 & 0 \\ px & 1 \end{pmatrix}\right]\left[\begin{pmatrix} 1 & ap^{m-1} \\ 0 & 1 \end{pmatrix}\right] = \left[\begin{pmatrix} 1 & p^{m-1}a \\ px & 1 \end{pmatrix}\right] = \left[\begin{pmatrix} 1 & p^{m-1}a \\ 0 & 1 \end{pmatrix}\right]\left[\begin{pmatrix} 1 & 0 \\ px & 1 \end{pmatrix}\right]$$

So that $\mathcal{M}$ is indeed preserved by $\mathbb{C}(I/I_m)$. $\square{-1}$
Claim 2: $[t]$, $[r]$ and $[s]$ preserve $\mathcal{M}$.

Proof of Claim 2: By the above computation, it is not hard to see that $(1 - v(bp^m))$ and $(1 - u(ap^{m-1}))$ commute, so that t will preserve $\mathcal{M}$. Finally, and most crucially

$$[s](1 - v(bp^m)) = 0$$
$$[r](1 - u(bp^{m-1})) = 0$$

So that again as our pre-factors commute, $\mathcal{M}$ is annihilated by r and s. □−2
$\mathcal{N}$ is a proper subspace of $\mathbb{C}(I/I_l)$ so our bound on dimension is immediate. □

With this, we prove our final Proposition. We forego the notation of [Ber74] for consistency with our past results of similar flavours.

Proposition 5.17. Suppose (π, V) is an irreducible, admissible representation of G admitting an I_m fixed vector. Then:

$$\text{Dim}_{\mathbb{C}}(V^I) \leq \begin{cases} 2 \times p^{4m-2}(p-1)^2 \text{ if } \mathcal{M}V^{I_m} \neq 0 \\ 32 \times p^{8m-8}(p-1)^4 \times (p - \frac{1}{2})^2 \text{ otherwise} \end{cases}$$

Proof. Write $W := V^{I_m}$. Then for each $w \in W$, $\mathcal{M}W$ is a $\mathcal{H}$-submodule of W. So it is trivial, or W.
Case 1 - ($\mathcal{M}W \neq 0$): Take $w \in W$ so that $\mathcal{M}w \neq 0$. This is again a $\mathcal{H}$-submodule; as it is non-zero, it must be W. Because $[t]^2$ is central, it acts as a scalar by Schur's Lemma. So:

$$\text{Dim}_{\mathbb{C}}(W) \leq \text{Dim}_{\mathbb{C}}(\mathcal{N}w \oplus \mathcal{N}[t]w) \leq 2\text{Dim}_{\mathbb{C}}(\mathcal{N})$$

and the bound produced in Proposition 5.16 completes the first claim.
Case 2 - ($\mathcal{M}W = 0$): We first note that as $\mathcal{M}$ is a left ideal, $\mathcal{MH}$ is a two sided ideal of $\mathcal{H}$. So certainly:

$$\mathcal{MH}W = \mathcal{M}W = 0$$

Meaning that W is a $\mathcal{H}/\mathcal{MH}$-module, which crucially will still be irreducible. We now seek to use Proposition 5.14 to bound the dimension of W. Now, in the quotient:

$$(1 - v(bp^m))(1 - u(ap^{m-1})) + \mathcal{MH} = 0,$$

which we rearrange as:

$$v(bp^m) + u(ap^{m-1}) - 1 + \mathcal{MH} = v(bp^m)u(ap^{m-1}) + \mathcal{MH}$$

Crucially, this tells us that the image of $\mathbb{C}(I/I_m)$ has decreased dimension. Writing each element of $\mathbb{C}(I/I_m)$ in the form of Equation 5.1:

$$\left[\begin{pmatrix} \lambda & \mu \\ p\pi & \nu \end{pmatrix}\right] = \left[\begin{pmatrix} \lambda & 0 \\ 0 & \nu - p\lambda^{-1}\mu\pi \end{pmatrix}\right] \left[\begin{pmatrix} 1 & 0 \\ -(\nu - p\lambda^{-1}\mu\pi)^{-1}p\pi & 1 \end{pmatrix}\right] \left[\begin{pmatrix} 1 & \lambda^{-1}\mu \\ 0 & 1 \end{pmatrix}\right]$$

Writing $\pi = \pi' + p^{m-1}a$ and $\mu = \mu' + p^{m-1}b$, we see

$$\left[\begin{pmatrix} 1 & 0 \\ p\pi & 1 \end{pmatrix}\right]\left[\begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}\right] + \mathcal{MH} = \left[\begin{pmatrix} 1 & 0 \\ p\pi & 1 \end{pmatrix}\right]\left[\begin{pmatrix} 1 & \mu' \\ 0 & 1 \end{pmatrix}\right] + \dots$$
$$\dots + \left[\begin{pmatrix} 1 & 0 \\ p\pi' & 1 \end{pmatrix}\right]\left[\begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}\right] - \left[\begin{pmatrix} 1 & 0 \\ p\pi' & 1 \end{pmatrix}\right]\left[\begin{pmatrix} 1 & \mu' \\ 0 & 1 \end{pmatrix}\right] + \mathcal{MH}$$

So that we may restrict our basis to matrices where one of π, μ is in $\{0, \dots, p^{m-1}-1\}$. Now, writing $N_x = \{0, \dots, x-1\}$ for $x \in \mathbb{Z}$, we define:

$$J := \left\{\left[\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\right] v(p\pi)u(\mu) : a, d \in \mathbb{Z}_p^\times,\ (\pi \in N_{p^{m-1}},\ \mu \in N_{p^m})\right\} \cup \dots$$
$$\dots \cup \left\{\left[\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\right] v(p\pi)u(\mu) : a, d \in \mathbb{Z}_p^\times,\ (\pi \in N_{p^m},\ \mu \in N_{p^{m-1}})\right\}$$

We calculate that $|J| = (p^m - p^{m-1})^2(2p^{2m-1} - p^{2m-2})$. So then, replacing I/I_m by J in our choice of $X_1, \dots, X_a, Y_1, \dots, Y_b$, we decrease our parameters to:

$$a = 4((p^m - p^{m-1})^2(2p^{2m-1} - p^{2m-2}),$$

$$b = 2(p^m - p^{m-1})^2(2p^{2m-1} - p^{2m-2}),$$

yielding the desired result. □

Remark 5.18. In this Chapter we have constructed all of the 1-dimensional representations of our algebras, and improved on bounds of Bernshtein, by a factor of approximately p^4. Moreover, we have given a criterion which when attained improves Bernshtein's bound by a factor of p^{4m+2}, and when failed, produces an annihilating ideal. This result concludes the dissertation.

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