§0 Introduction / Aims

This talk is a follow on from the talk JC recently gave on sheaves, both of these being a primer for the reading group JC would like to start next term on Condensed Mathematics. Condensed Mathematics in some sense resolves that the category of topological abelian groups isn't an Abelian Category. It does this by looking at certain sheaves on a site.

The goal of todays talk is to introduce the definition of sites in a way that makes them seem relatively obvious, and put them into historical context.

I'm very sorry but I have put this talk together in a bit of a rush, so that it can be done before the easter holiday!

§1 The Weil Conjectures (Motivation)

In 1949 Andre Weil published "Numbers of Solutions of Equations in Finite Fields", a paper where he laid out a series of conjectures, which became known as the Weil Conjectures.

Suppose we have a variety X defined over a finite field k; for the moment, this means we have a set of polynomials $f_1, \dots, f_m \in k[x_1, \dots, x_n]$ so that

$$X = \{(u_1, \dots u_n) \in k^n : f_i(u_1, \dots, u_n) = 0 \ \ i = 1, \dots, m\}$$

Then, taking K_s to be the unique extension of k of degree $|k|^s$, we can obviously include $k[x_1, \dots, x_n] \to K_s[x_1, \dots, x_n]$, and write

$$X_s = \{\{(u_1, \dots u_n) \in K_s^n : f_i(u_1, \dots, u_n) = 0 \ \ i = 1, \dots, m\}\}$$

Which is, the set of points satisfying our polynomials whose coordinates lie in some bigger field. Weil then defined a function

$$f(t) = \exp\left(\sum_{s \geq 1} |X_s| \frac{t^s}{s}\right)$$

which is often called the *zeta function of X*. Weil then made 4 conjectures, each of which turned out to be true. For now I will state the first two conjectures:

1. $f(t)$ is a rational function in t, so we may write $f(t) = p(t)/q(t)$
2. $f(t)$ has a certain nice factorisation, (by which I mean) $p(t)$ and $q(t)$ admit nice factorisations.

I think what is most striking about these, is he apparently had incredibly small amounts of numerical evidence for his claims. In fact, the seed of many of the conjectures come from analogues with classical theorems of algebraic topology. Notably, if we write

$$\bar{X} = \bigcup_{s \in \mathbb{N}} X_s \subset \bar{k}^n$$

We have an automorphism of $\bar{k}^n$ given by

$$\varphi : x \mapsto x^p$$

called the Frobenius automorphism. It turns out that

$$X_s = \text{Fix}(\phi^s) = \{x \in \bar{X} : x = \varphi^s(x)\}$$

We can now state (nearly) the celebrated Lefschetz fixed point theorem:

Theorem Let X be a "nice" topological space, and $g : X \to X$ a continuous map, which is "nice". Then we get maps

$$H^i(g) : H^i(X; \mathbb{Q}) \to H^i(X; \mathbb{Q})$$

which satisfy

$$\# \operatorname{Fix}(g) = \sum_{i \in \mathbb{N}} (-1)^i \operatorname{tr}(H^i(g))$$

So then, one might hope that we could infer information about our zeta function $f(t)$ by having a reasonable theory of cohomology for varieties.

As Weil was motivated by these deep ideas in cohomology, he suggested that the solution of these problems might come from a "nice" theory of cohomology; one that was lacking for varieties. It turns out, these both follow from a nice notion of cohomology:

- Point 1 is immediate from finite dimensionality of certain cohomology groups
- Part 2 is a consequence of "Poincare Duality" of our nice cohomology; Grothendieck found an analogue of the Lefschetz fixed point formula to prove this.

§2 Sites / Grothendieck Topologies

Sheaves

At this point, I want to quickly recall from JC's talk, what it means to be a sheaf; for now, take X a topological space, and recall that we have a natural category

$$\mathcal{O}(X) = \begin{cases} \text{objects: open subsets} \\ \text{morphisms: inclusions of open subsets} \end{cases}$$

Definition A sheaf on X taking values in a category $\mathcal{A}$ is a functor $\mathcal{F} : \mathcal{O}(X)^{op} \to \mathcal{A}$, satisfying, for $U \subset X$ open, and an open covering of U, $\{U_i\}$:

$$\mathcal{F}(U) \longrightarrow \prod \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \cap U_j)$$

that the preceding diagram is an equaliser. This definition is quite brief; in short it tells us:

1. I can glue together elements in $\mathcal{F}(U)$ from collections of elements of $\mathcal{F}(U_i)$ that agree on intersections
2. My gluing is unique

Sheaves are Bad

I've said that these problems could be resolved with a nice theory of cohomology; you might be screaming at me, we have lots of nice theories of cohomology! Can't we for instance, use singular cohomology, or sheaf cohomology? The answer, unfortunately is no:

Lemma If Y is an irreducible algebraic variety, then $H^*(Y, \mathcal{F}) = 0$ for any locally constant sheaf $\mathcal{F}$ on Y.
Proof The sheaf of locally constant functions on an irreducible variety is in fact constant, as any two open sets in an irreducible variety. Thus the sheaf is flabby, so some basic sheaf theory gives the result.

So we need to find some other way of doing cohomology, that somehow fixes whatever's going wrong with our algebraic varieties. The problem is that the Zariski topology doesn't have enough open sets; our sheaf condition isn't strong enough and we don't get enough information about cohomology. So we need some way of "adding" more open sets to our sheaf condition.

Our definition of sheaf very conveniently seems to begging for us to do this; our sheaf is defined as a functor on our open subsets ($\mathcal{O}(X)$), we simply want to look at a bigger category!

So then, now we want to define what it means to be a sheaf on some category $\mathcal{C}$; looking back at our definition of sheaves on topological spaces, it better well be a functor $\mathcal{F} : \mathcal{C}^{op} \to \mathcal{A}$ that satisfies some equaliser condition. The only thing that isn't obviously generalisable, is the notion of open cover.

Open covers

We want to generalise the idea of an open cover, so we should investigate what properties they have in the category $\mathcal{O}(X)$. First I'm going to make some uncontroversial statements about covers, and them I'm going to try to translate them into more categorical language.

1. *(Form of a cover)* A cover of $U \in \mathcal{O}(X)$ is a collection of open sets $U_i \subset U$ *(I do not mean to say this is sufficient)*
2. *(Trivial Cover)* U is a cover of U
3. *(Intersection Stability)* If I have an open cover $\{U_i\}$ of U, and $V \subset U$, then $\{U_i \cap V\}$ is an open cover of V
4. *(Local Determination)* If I have an open cover $\{U_i\}_{i \in I}$ of U, and for each $i \in I$ an open cover $\{V_{i,j}\}_{j \in J_i}$ of U_i, then the collection $\{V_{i,j}\}_{j \in J_i}$ is an open cover of U.

To begin to translate these, we're going to need a categorical dictionary:

Topological	Categorical
Inclusions $V \subset U$	Morphisms $V \to U$
Intersections $V \cap W \subset U$	Pullbacks $V \times_U W \to U$

To justify this last one, consider the diagram below in $\mathcal{O}(X)$:

$$\begin{tikzcd} & W \arrow[d] \\ V \arrow[r] & U \end{tikzcd}$$

The pullback of this diagram (if it exists) will be an object , so that if X includes into W and V, we have

$$\begin{tikzcd} X \arrow[dr] & & \\ & V \times_U W \arrow[r] \arrow[d] & W \arrow[d] \\ & V \arrow[r] & U \end{tikzcd}$$

But obviously, this says that $X \subset U$ and $X \subset W$ iff $X \subset U \cap W$. So then, translating our statements to a category C:

1. *(Form of a cover)* A cover of $C \in \mathcal{C}$ is a collection of morphisms $\{f_i : C_i \to C\}$ *(I do not mean to say this is sufficient)*
2. *(Trivial Cover)* If $C' \to C$ is an isomorphism, $\{C' \to C\}$ is a cover of C.
3. *(Intersection Stability)* If I have a cover $\{f_i : C_i \to C\}$ of C, and $g : D \to C$, then the collection of maps $\{C_i \times_C D \to D\}$ induced by pullbacks is a cover of D
4. *(Local Determination)* If I have a cover $\{f_i : C_i \to C\}$ of C, and for each $i \in I$ a cover $\{g_{i,j} : C_{i,j} \to C_i\}$ of C_i, then the collection $\{f_i \cdot g_{i,j} : C_{i,j} \to C\}$ is a cover of C.

This is technically a coverage, and not a site
Definition A Grothendieck topology* $\mathcal{J}$ on a category $\mathcal{C}$ with pullbacks is, for every object $C \in \mathcal{C}$ a collection of covers of C, which we denote $J(C)$, satisfying the axioms above.
A category $\mathcal{C}$ equipped with a Grothendieck topology is called a site.

Remark Really here, I am defining a coverage; defining a Grothendieck topology involves talking about sieves, which I wanted to avoid. They're not too difficult to understand, they're just fiddly.

This notion of "covering" turns out to be exactly what was needed to get a notion of sheaves where we've added "more open sets". I will describe the category of that Serre used to define the étale cohomology soon, but first I quickly want to show how we define sheaves on a site:

Definition A sheaf on a site $(\mathcal{C}, \mathcal{J})$ (that is, a category $\mathcal{C}$ equipped with a Grothendieck topology $\mathcal{J}$) taking values in a category $\mathcal{A}$, is a functor $\mathcal{F} : \mathcal{C}^{op} \to \mathcal{A}$, so that for $C \in \mathcal{C}$ and a collection of morphisms $\{f_i : C_i \to C\} \in \mathcal{J}(C)$, the diagram

$$\mathcal{F}(C) \longrightarrow \prod \mathcal{F}(C_i) \rightrightarrows \prod_{i,j} \mathcal{F}(C_i \times_C C_j)$$

is an equalizer.

§4 Weil Conjectures Solved!

I will briefly say a few words on the site used to define étale cohomology; the cohomology that was (in part) used to solve the Weil Conjectures. Take a variety X, and form

$$\acute{E}t(X) = \begin{cases} \text{objects: } E \to X \text{ étale} \\ \text{morphisms: compatible triangles of morphisms} \end{cases}$$

Before you ask: I could tell you what étale means, but it's not too enlightening; they are essentially "local isomorphisms", but because of the Zariski Topology being a bit awful, we have to do some extra work to define exactly what we mean.

A collection of morphisms in this category is called a cover if they are jointly surjective; that is the union of their images form the whole set. So then, a cover of $X \to X$ is a collection of étale maps $E_i \to X$, whose images jointly cover X. The next obvious question is how we find our cohomology groups from this:
Lemma Take $\mathcal{C}$ a site. The category of abelian sheaves on $\mathcal{C}$, $\mathtt{Ab}(\mathcal{C})$ is an abelian category.

So then the cohomology of a sheaf is given by the considering derived functors of evaluation at X.

Cool stuff about sites

Now that I've covered what the talk was supposed to be about, I want to say some cool things about Sites. I'm sorry if you've heard any of this before, because I have been ranting about this stuff for a little bit. Most of these examples are taken from a talk given by Scholze.

More on Étale Maps

For a scheme X we have (roughly) defined the Étale site of X. If k is a field, then

$$X = \text{Spec}(k)$$

then

$$\acute{E}t(X) = \{f : Y \to \text{Spec}(k) \text{ étale}\}$$

It turns out that, to be an étale bundle over a field

$$Y = \coprod \text{Spec}(K_i)$$

Where each K_i is a finite separable extension of k. Now, by Galois Theory, this is

$$\acute{E}t(X) \cong \{\text{ sets with a continuous action of } \text{Gal}(\bar{k}/k)\}$$

(Remark for Matt) Supposedly to get this, you send Y to the $\bar{k}/k$ valued points of Y.

Profinite Cohomology

If G is a profinite group, it's reasonable to try to do group cohomology, while trying to take into account the topological nature of the group. We can consider

$$\mathcal{C} = G\text{-sets} = \{\text{ sets with a continuous } G\text{-action}\}$$

Where our covers are jointly surjective maps. We could further consider

$$\mathcal{D} = G\text{-pfsets} = \{\text{profinite sets with a continuous } G\text{-action}\}$$

Where our covers are again jointly surjective maps, but we now ask that if

$$\{f_i : S_i \to S\}_{i \in I}$$

is a cover of S, then a finite subset of I covers S. Then, if we look at Abelian sheaves on this latter site, any G module gives rise to an abelian sheaf, and taking cohomology we get group cohomology.

Grothendieck Toposes

I wanted to write about Grothendieck Toposes, but I ran out of time! :-)