1 What is Hoschild Cohomology?

Let k be an associative, commutative, unital ring, and let A be an associative k-algebra; that is, we require that A is a k-module, and that multiplication is k-bilinear.

Definition 1 (Opposite and Enveloping Algebra)**.** *Define A^{op} to be the k-algebra with underlying k-module A, and multiplication defined by*

$$a \star b = b \cdot a$$

Define the enveloping *algebra of A to be $A^e := A \otimes A^{op}$, which is endowed with the structure of a k-algebra via multiplication*

$$a_1 \otimes b_1 \cdot a_2 \otimes b_2 = a_1 a_2 \otimes b_2 b_1$$

By an A-bimodule, we mean a a k-module M that is both a left and right A-module, whose left and right actions commute.

Lemma 1. *An A-bimodule is exactly an A^e-module.*

Hochschild (Co)homology arises as the (co)homology of the Barr Resolution, and has a nice interpretation when our Algebra A is nice over k. Before we dive in to the precise definition of Hochschild (co)homology, I'll say what this is:

Definition 2 (Hochschild (Co)Homology for nice A)**.** *When A is projective over k, we define the Hochschild (Co)Homology of an A-bimodule M to be*

$$HH_n(A, M) \cong \mathrm{Tor}_n^{A^e}(M, A)$$

$$HH^n(A, M) \cong \mathrm{Ext}_{A^e}^n(M, A)$$

[Bar Complex of A] So for instance, if we were to consider the Hochschild Cohomology of a group ring, we would be in this instance.
When A is not projective (or flat in the case of Homology) over k, this is not quite the case. To get the more general version, we define the *Bar Complex*

Definition 3. *For A, k as defined earlier, define the complex $C_\star$ of A-bimodules:*

$$\ldots \longrightarrow A^{\otimes 4} \longrightarrow A^{\otimes 3} \longrightarrow A \otimes A \xrightarrow{\pi} A \longrightarrow 0$$

Where the map π is multiplication, and the map $d_n : A^{\otimes n} \to A^{\otimes n-1}$ is defined via

$$d_n(a_0 \otimes \ldots \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_{n+1}$$

Each degree of the chain complex becomes an A-bimodule by multiplication on the first and last factors, under which each d_n becomes a bimodule-map. These differentials make $C_\star$ an exact complex, which can be seen by constructing a section.
Then, we write $B_\star(A)$ for the truncated complex ending at $A \otimes A$, which we call the Bar Complex *of the A-bimodule A.*

Definition 4 (Hochschild Chains and Cochains)**.** *Let M be an A-bimodule. We define two new complexes*

$$C_\star(A, M) = M \otimes_{A^e} B_\star(A)$$

$$C^\star(A, M) = \mathrm{Hom}_{A^e}(B_\star(A), M)$$

which we call the Hochschild Chains *and* Hochschild Cochains *respectively.*

Remark 1. According to n-lab, these are in fact the regular tensor product, and Hom functors in the symmetric monoidal $(\infty, 1)$-category of chain complexes - isn't that cool!

Definition 5 (Hochschild (Co)Homology; for real this time)**.** *We define the Hochschild (Co)Homology of an A-bimodule M to be*

$$HH_n(A, M) = H_n(C_\star(A, M))$$

$$HH^n(A, M) = H^n(C^\star(A, M))$$

Remark 2. It should be obvious then that this will turn out to be Ext and Tor when the objects $A^{\otimes n}$ are projective (or for Homology, it's enough to be flat) over A^e - this need not always be the case, say when k embeds into A in some pathological way.

As an A^e-module, we might think of $A^{\otimes n} \cong A^e \otimes_k A^{\otimes(n-2)}$ (compatible with the action of A^e); then

$$\mathrm{Hom}_{A^e}(A^{\otimes n}, -) \cong \mathrm{Hom}_{A^e}(A \otimes A^{\otimes n-1}, -) \cong \mathrm{Hom}_k(A, \mathrm{Hom}_{A^e}(A^{\otimes n-1}, -))$$

So by induction, we see that projectivity over k gives us projectivity over A^e. For flatness, we take an exact sequence, which must be an exact sequence of k-modules. Then tensoring by any factor preserves exactness, so repeating the process we see that the combined tensor preserves exactness, so our module is flat.

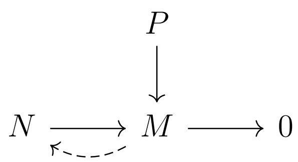
2 Simplicial Methods

The Bar complex actually arises from deeper structure, contained in the data of a k-algebra. Firstly we note the map $i : k \to R$ gives rise to functors

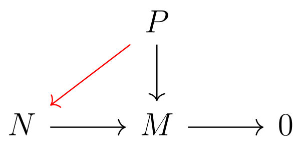
$$k - mod \underset{R\otimes_k -}{\overset{i^*}{\rightleftarrows}} R - mod$$

where in fact $(R \otimes_k -) \dashv i^*$, a property of restriction of scalars. Notably then, the endofunctor $R \otimes_k --$ on $R - mod$ is a comonad. It turns out that, the structure of a comonad is enough to define a simplicial set for each element of an abelian category; these simplicial sets give rise to chain complexes - which in fact are our Bar Complexes. We might then define these as our resolutions of interest, and proceed with our theory of Homology and Cohomology. We can however do more.

Definition 6. *For a map of rings $k \to R$, we call $P \in R - mod$ $\otimes$-projective when, for any diagram*



(where the dashed arrow indicates the map $N \to M$ is k-split) there exists a lifting



so that the diagram commutes.

This is equivalent to the following:

Proposition 1. With the setup above, A is $\otimes$-projective iff the map

$$R \otimes A \to A$$

admits a section

Of course, by the axioms of a comonad, the co-unit of the adjunction provides a map

$$R \otimes A \to R \otimes R \otimes A$$

So that all objects of the form $R \otimes A$ are $\otimes$-projective; of course, A need only be a k-module, not even an R-module. This can be useful for explicitly creating resolutions.

3 Ok, why do we care?

3.1 Group Cohomology

Recall 1 (Group Cohomology)**.** *Given a group G and a ring k, we form an associative algebra kG by endowing the free k-module on G with multiplication from the group. Then given a kG-module, we form the "group cohomology" of G with coefficients in M as the right derived functor of the invariants $-^G$; this tells us information about the group, which of course is in some sense directly information about the algebra.*

We have the following proposition:

Lemma 2. *We have adjoint functors*

$$kG - mod \underset{L}{\overset{R}{\rightleftarrows}} kG - bimod$$

$L \dashv R$, with R exact (and so L preserves projectives).

Proof. The functor $R : kG - bimod \to kg - mod$ takes $M \; inkG - bimod$ to the kG-module M whose G action is defined by

$$g \star m = gmg^{-1}$$

L takes the kG-module M to the kG-bimodule $M \otimes_k kG$, with left action: $g \star m \otimes h = gm \otimes gh$ and right action $m \otimes h \star g = m \otimes hg$ it's an interesting exercise to show that these are in fact adjoint. □

Proposition 2. We have

$$H^n(G; RM) \cong HH^n(kG, M)$$

As clearly, any kG-module M may be turned into a kG-bimodule admitting a trivial right action, we see that Hoschild Cohomology and Group Cohomology agree over the group ring.

Proof. As L is left adjoint to an exact functor, it preserves projectives. Now taking $P_\bullet \to \mathbb{Z}$ a projective resolution

$$\begin{aligned} H^n(G; CM) &\cong \mathrm{Ext}^n(\mathbb{Z}, CM) \\ &\cong H^n(\mathrm{Hom}(P_\bullet, CM)) \\ &\cong H^n(\mathrm{Hom}(LP_\bullet, M)) \\ &\cong \mathrm{Ext}^n(kG, M) \cong HH^n(kG, M) \end{aligned}$$

As required. □

So then, it appears that Hochschild Cohomology is a generalisation of group cohomology, to associative algebras over a ring.

3.2 Block Theory

Take a field k and a finite group G. Then we can write

$$1 = e_1 + e_2 + \ldots + e_n$$

where each e_i has that $e_i kG$ is an indecomposable algebra. Then we can write

$$kG = \bigoplus_{i=0}^{n} e_i kG$$

Which will distribute through to Hochschild Cohomology:

$$HH^n(kG, kG) = \bigoplus_{i=0}^{n} HH^n(kG, e_i kG)$$

In some sense, the regular cohomology ring with coefficients in k is the trivial block here, so that Hochschild Cohomology sees more information about the blocks than the regular group cohomology ring.

4 Examples

If $A = k[x]$, for k a field, we can write $A^e \cong k[x, y]$, then we have a short exact sequence:

$$0 \longrightarrow k[x,y] \xrightarrow{x-y} k[x,y] \xrightarrow{\pi} k[x] \longrightarrow 0$$

we can see this is exact as the kernel of the map π must be of height one, and contain an irreducible element. This is enough to show that the kernel is generated by this element, and the kernel contains $x - y$. This yields a projective resolution $C_\bullet$

$$0 \longrightarrow k[x,y] \xrightarrow{x-y} k[x,y] \longrightarrow 0$$

So that, taking an A-bimodule M,

$$HH_n(A, M) = H_n(C_\bullet \otimes M)$$

and

$$HH^n(A, M) = H^n(Hom(C_\bullet, M))$$

So we can calculate in both cases:

$$HH_0(A, M) = M/(xM - Mx) = HH^1(A, M)$$
$$HH_1(A, M) = \{m \in M : mx = xm\} = HH^0(A, M)$$

and the (co)homology is zero in all other degrees.

5 Low Degree Hochschild Cohomology

Recall, as was said earlier, we have endowed $A^{\otimes n}$ with the structure of an A-bimodule by multiplying on the left in the first factor, and multiplying on the right in the last factor. This tells us that

$$A^{\otimes n} \cong A^e \otimes A^{\otimes n-2}$$

We're now going to try to compute the low degree Hochschild Cohomology groups, and see what they tell us. Take an A-bimodule M. We get a complex

$$\dots \mathrm{Hom}_{A^e}(A^{\otimes 4}, M) \longleftarrow \mathrm{Hom}_{A^e}(A^{\otimes 3}, M) \longleftarrow \mathrm{Hom}_{A^e}(A \otimes A, M) \longleftarrow 0$$

Which we might write:

$$\dots \mathrm{Hom}_{A^e}(A^e \otimes A^{\otimes 2}, M) \longleftarrow \mathrm{Hom}_{A^e}(A^e \otimes A, M) \longleftarrow \mathrm{Hom}_{A^e}(A^e, M) \longleftarrow 0$$

which then by tensor-hom becomes:

$$\dots \mathrm{Hom}_k(A^{\otimes 2}, M) \longleftarrow \mathrm{Hom}_k(A, M) \longleftarrow \mathrm{Hom}_k(k, M) \longleftarrow 0$$

So now we just need to work out the form of the maps. We know for $\phi \in \mathrm{Hom}_{A^e}(A^e \otimes A^{\otimes n-2}, M)$ we have that

$$\begin{aligned} d\phi(a_0 \otimes \dots a_n) &= \phi(d(a_0 \otimes \dots \otimes a_n)) \\ &= \sum_{i=0}^{n} (-1)^i \phi(a_0 \otimes \dots \otimes a_i a_{i+1} \otimes \dots a_n) \\ &= a_0 a_1 \phi(1 \otimes a_2 \otimes \dots \otimes a_{n-1} \otimes 1) a_n + \dots \\ \dots + &\sum_{i=1}^{n-1} (-1)^i a_0 \phi(1 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes 1) a_n + \dots \\ \dots + &(-1)^n a_0 \phi(1 \otimes a_1 \otimes \dots \otimes a_{n-2}) a_{n-1} a_n \end{aligned}$$

We can use this to work out our Hochschild Cohomology.
Degree 0: The 0th Cohomology is the kernel of the map

$$d : \mathrm{Hom}(k, M) \to \mathrm{Hom}(A, M)$$

defined so that

$$\begin{aligned} d\phi(a) &= d\phi(1 \otimes a \otimes 1) \\ &= a\phi(1) - \phi(1)a \end{aligned}$$

So then, associating $\mathrm{Hom}(k, M)$ with M, we see that

$$HH^0(A, M) = \{m \in M : am = ma \text{ for all } a \in A\}$$

Degree 1: The 1st Cohomology is $\mathrm{Ker}(d_1)/\,\mathrm{Image}(d_0)$. We have just calculated:

$$d\phi(a) = a\phi(1) - \phi(1)a$$

So that maps

$$\mathrm{Image}(d_0) = \{f : A \to M : \exists a \in A \text{ such that } f(x) = ax - xa\} =: \mathrm{PDer}_k(A, M)$$

We sometimes call such maps "principle derivations".
So now we only need to calculate d_1.

$$\begin{aligned} d(\phi(a \otimes b)) &= \phi(d(1 \otimes a \otimes b \otimes 1)) \\ &= \phi(a \otimes b \otimes 1 - 1 \otimes ab \otimes 1 + 1 \otimes a \otimes b) \\ &= a\phi(b) - \phi(ab) + \phi(a)b \end{aligned}$$

So then, the condition that $\phi \in \mathrm{Ker}(d_1)$ is that

$$\phi(ab) = a\phi(b) + \phi(a)b$$

that is, we ask that ϕ is a derivation of A over k. The set of these is normally denoted

$$\mathrm{Der}_k(A, M) := \{\phi : A \to M : \phi(ab) = a\phi(b) + \phi(a)b\}$$

Degree 2: Now we calculate $d_2(\phi)$

$$\begin{aligned} d_2(\phi)(a \otimes b \otimes c) &= \phi(d_2(1 \otimes a \otimes b \otimes c \otimes 1)) \\ &= a\phi(b \otimes c) - \phi(ab \otimes c) + \phi(a \otimes bc) - \phi(a \otimes b)c \end{aligned}$$

So then the 2-cocycles are functions $\phi : A \otimes A \to M$ so that

$$\phi(ab \otimes c) - \phi(a \otimes bc) = a\phi(b \otimes c) - \phi(a \otimes b)c$$

while the 2-coboundaries correspond to maps $f : A \otimes A \to M$ so that there is $g : A \to M$ with

$$f(a \otimes b) = ag(b) - g(ab) + g(a)b$$

It turns out, the 2-cocycles we just worked out correspond to certain algebra structures on the A-bimodule $A \oplus M$, and the coboundaries correspond to "trivial extensions"; so then, each element of $H^2(A, M)$ corresponds uniquely to one of these extensions.

Definition 7 (Square Zero Extension)**.** *A square zero extension of A by I is a SES of k-modules*

$$0 \longrightarrow I \longrightarrow E \xrightarrow{\phi} A \longrightarrow 0$$

so that E is a k-algebra, ϕ is a k-algebra morphism, and I is an ideal of E so that $I^2 = 0$. This endows I with the structure of an A-bimodule, via

$$E/I \cong A$$

We call this a Hochschild Extension when it is k-split, so that the SES is split as a sequence of k-modules. There is an obvious notion of equivalence of these extensions.

Theorem 1. *Hochschild Extensions of A by M are in 1-1 correspondence with $HH^2(A, M)$.*

6 Cohomology Products