Hello everyone, I'm Matt, welcome to your first lesson in Advanced Hole Counting. We're going to start by going over some of the ideas you might have gone over in Hole Counting 101, and then progress onto some more difficult examples. To begin, I'd like to state a motivating problem that we're going to keep coming back to throughout the course of the lecture:

> **Problem:** Alice owns an apple orchard, and Bob would like to buy some of her apples. Bob only wants to buy apples that worms haven't eaten all the way through. How can he tell whether an apple meets his criteria?

Hopefully this problem can remind us that what we're doing actually has practical relevance. We're going to consider this in three contexts:

1. The apple is a finite simplicial complex (perhaps Bob and Alice live in a PS2)
2. The apple is a topological space admitting a finite cover of contractible open sets
3. The apple is $\mathbb{F}_5[X, Y, X^{-1}, Y^{-1}]$

In earnest, what I want to do is chart the course through the historical development of Algebraic Topology, especially Čech Homology, which I think is really genius. Then I want to talk about how to take this idea of Homology, and show how it naturally extends to rings (what the hell!).

1 Finite Simplicial Homology

1.1 Definitions

If any of you have ever done any hole counting (which also goes by "algebraic topology") before, then you'll have probably have seen this and this will be recap. Because I don't want to get bogged down in the details, I'm going to go over this sketchily.

Definition 1 (n-Simplex)**.** *This is going to be a definition by example;*

0. a 0-simplex is a point

1. a 1-simplex is a line

2. a 2-simplex is a triangle

3. a 3-simplex is tetrahedron

4. ...

and the higher simplices are higher dimensional generalisations of these.

You might be yearning for a more precise definition of the higher simplices are - it turns out, this is very easy to give, but just not very enlightening.

Definition 2 (Δ-set)**.** *A Δ-set is a topological space, made by gluing together a finite number of simplices along their edges, along with the data of the gluing.*

Example 1 (Examples). *1. A point*

2. A circle glued out of 2 1-simplices

3. a tetrahedron with a "stem" sticking out the top.

Example 2 (Non-examples). *A tetrahedron with a "stem" sticking out of the middle of a face.*

1.2 What is a hole?

With these sketchy definitions in mind, we can move on to how we count holes. We first take a simplicial complex V, and write down the sets

$$V_n := \{ \text{ the n-simplices that make up } V\}$$

for each $n \in \mathbb{N}$, and also maps

$$d_i : V_n \to V_{n-1}$$

sending each n-simplex to its i^{th} "face" (which is the face opposite the i^{th} vertex). Then we construct a "chain complex"

$$\dots \longrightarrow \mathbb{Z}V_4 \longrightarrow \mathbb{Z}V_3 \longrightarrow \mathbb{Z}V_2 \longrightarrow \mathbb{Z}V_1 \xrightarrow{d_o} \mathbb{Z}V_0 \longrightarrow 0$$

where in the i^{th} spot we put the free abelian group on the set of n-simplices of V, and we define maps going down by sending an n-simplex to it's boundary (with certain signs), and extending "linearly" to the rest of the Abelian group.

The idea is, that if I have a hole, I can draw a loop around it that I can't shrink. Also, loops I can shrink are basically those that arise as the boundary of some shape. So then, to find holes, I want to find loops, which don't come from the boundary of a shape. Now, we said that the d maps send a simplex to its boundary, so then

$$\text{“Loops coming from boundaries”} = \text{Im}(d_1)$$

and the "boundary with signs" of an edge is

$$d_0(\text{an edge}) = \text{ (its start point)} - \text{(its end point)}$$

and then

$$\text{“Loops”} = \text{Ker}(d_0)$$

so then, our apple has no holes when

$$\text{Im}(d_1) = \text{Ker}(d_0) \iff \text{Ker}(d_0)/\text{Im}(d_1) = 0$$

So then, the quotient on the right vanishes when we have no holes, and in some sense "counts holes" when it's not zero. So we now know how to count holes in a finite simplicial set.

Example 3 (Cute Apple). *(Tetrahedron with a stick poking out).*

Example 4 (Holey Apple). *(Triangular prism with faces missing)*

2 Čech Homology

2.1 Hold on... I've seen this before...

This is all well and good, but we should be honest with ourselves and recognise that in the real world, apples frequently don't appear as simplicial complexes. A more general example is that of a topological space, which doesn't in general have the structure of a simplicial complex. Now in a first course on hole counting, you might have seen what's called singular homology, where we make another chain complex, with objects

$$\mathbb{Z}(\mathrm{Cont}(\Delta^i, X))$$

and maps defined in whacky ways. This is in practice, completely useless to calculate with, so we'd like to do something better. Here, Čech Cohomology comes in.

2.2 Čech Cohomology

Suppose you're a mathematician in the 1920's, and you've seen Simplicial Homology, but you're not really satisfied. You'd quite like to do this on any topological space - but we know there's a problem, not every topological space can be conveniently triangulated. But also, lots of people have done a lot of work on simplicial homology, so it would be pretty great if you could just, you know, keep using it. You get a genius idea:

Definition 3 (Associated Simplicial Complex)**.** *Let T be a topological space, and $\mathcal{U} = \{U_i : i \in I\}$ be an open cover, (so that is, each of the U_i are open, and their union is T). The idea is to make a simplicial complex out of this open cover - if we draw our space on the left*

1. *For every open set U_i, we get a point i in the associated simplicial complex*
2. *Whenever two open sets U_i and U_j intersect in T, we draw an edge between i and j - (really, we draw as many edges between them as there are connected components)*
3. *Whenever three open sets intersect, we draw a triangle*

$\vdots$

We write this $\Delta(T, \mathcal{U})$. (Now technically, we wanted our simplicial set to have some underlying order on the vertices, so I think you may want this here)

I said this was a genius idea, but whats the payoff? The idea is that we can now use our techniques of simplicial homology on any topological space, if you're given an open cover.For now we're also going to make one further definition:

Definition 4 ((Nearly)-Good Cover)**.** *Let T be a topological space, and $\mathcal{U} = \{U_i : i \in I\}$ an open cover. We call the cover (nearly)-Good when $\mathcal{U}$ is finite, and each open, and all finite intersections of opens have contractible connected components.*

Definition 5 (Čech Cohomology)**.** *Take a space T (for which a nearly-Good open cover exists), as well as a (nearly)-Good open cover $\mathcal{U}$. The Čech Cohomology with coefficients in $\mathbb{Z}$ of T is*

$$\check{H}(T) := H^{\Delta}(\Delta(T, \mathcal{U}))$$

Remark 1. I've said that this is "*the* Čech Cohomology" but a priori, this Homology relies on the choice of (nearly)-Good open cover. This as you might expect, is because the homology is the same for any choice of good cover.

Remark 2. We actually take the cohomology of the simplicial complex - why we do this doesn't really matter

Proposition 1. The Čech Homology of a space is independent of choice of good cover.

Don't worry, we will have at least one proof in this talk.

Corollary 1. *Let S be a simplicial complex, (where for simplicity we require that no two simplices have the same vertex set). The Čech Homology of a simplicial complex is a simplicial complex.*

Proof. Rapidly, take a simplicial complex S, and a vertex s. Define the simplicial star of s to be

$$\text{Star}(s) := \bigcup_{\sigma : s \in \sigma} \sigma$$

Example 5. Some example of a simplicial star

Now the sets $\text{int}(\text{Star}(s))$ form an open cover of the simplicial complex. I'm quickly going to justify two facts:

1. Each simplicial star is contractible
2. Each intersection of simplicial stars is contractible

For the first point, we can note that every simplex is contractible to any vertex, and the key point is we can perform these contractions in compatible ways along the gluings. For the second, either the intersection of two stars is empty, or there is an edge between the two vertices. In the latter case, we can contract each simplex to the middle of the unique edge in a coherent way.

Now then, to get the Čech Cohomology, we have to draw a vertex for each open set - but this is each vertex. Two simplicial stars intersect if and only if there is an edge between the vertices, three of them intersect is there is a two simplex and so on. So we see that the simplicial complex we recover is the one we began with. □

So then Čech homology agrees with regular homology on simplicial sets, and if you know any algebraic topology, it agrees with singular homology on locally contractible paracompact spaces. (That's for you John).

Example 6. *Klein bottle with a hole*

2.3 B ut what's happening?

Ultimately, we're in the business of taking a topological space, and making a chain complex. Passing to a simplicial complex in some sense obscures what's happening - we make a simplicial complex out of the topological space, and then we make a chain complex out of it. Suppose I have an open cover, indexed by I $\{U_i : i \in I\}$, if I skip the over the step of making our simplicial complex, what we've really made is:

$$\ldots \longleftarrow \prod_{i,j,k\in I} C(U_i \cap U_j \cap U_j, \mathbb{Z}) \longleftarrow \prod_{i,j\in I} C(U_i \cap U_j, \mathbb{Z}) \longleftarrow \prod_{i\in I} C(U_i, \mathbb{Z}) \longleftarrow 0$$

Where, in the first place (which are my "points"), because all of my opens were connected, I just get a copy of $\mathbb{Z}$ for each connected component. Then, for each connected component of each intersection, I get a copy of $\mathbb{Z}$, which are my "edges". So here's the template of what we did:

1. Choose a nice open covering of our topological space
2. Make a chain complex, by looking at some nice breed of function from the open sets
3. Take cohomology!

3 Local Cohomology

3.1 The Reason I actually wanted to do the talk

I think personally that we've done quite well for ourselves here, and essentially we should be able to count any kind of topological holes that we want. But Bob now faces a new problem... Alice has given him a new apple, R, where R is a finitely generated algebra over an algebraically closed field k. To understand what this means, we're first going to have to go over some algebraic geometry. A lot of this is a bit complex, so I hope you'll believe me on a lot of this stuff:

3.2 A brief aside into algebraic geometry

At this point, anything you don't understand should be vibed out - I'm not going into the precise details here, but it is a beautiful subject. From now on, let k be an algebraically closed field;

Definition 6. *We call a subset V of k^n a variety when it is the vanishing locus of a collection of polynomials. Algebraically, that is, there are polynomials $f_1, \ldots, f_n$ so that*

$$V = \{(a_1, \ldots, a_n) \in k^n : f_i(a_1, \ldots, a_n) = 0\}$$

Example 7. *So then, the "complex" circle*

$$\{(x, y) : x^2 + y^2 = 1\} \subset \mathbb{C}^2$$

is a variety, as are things like quadratics or cubics.

Given a variety, we can ask how polynomial functions in $k[x_1, \ldots, x_n]$ act on it. Notably, even though two polynomials might be different, their difference might be zero on the variety, so that we can view them as the same polynomial function on our variety.

Definition 7. *For a variety V, the coordinate ring of V is defined to be the ring of polynomial functions on V, where we view two polynomials to be the same when they agree on V*

Remark 3. Very, very beautifully, it turns out that all finitely generated reduced k-algebras can be realised as coordinate rings as a variety, and that studying these rings from the geometric side can tell you very non-trivial facts about the rings. This fact is the basis for all of classical algebraic geometry.

It turns out that we can define some strange topology on any variety called the Zariski Topology, which we'll describe by their open sets:

Definition 8 (Affine Opens)**.** *Let V be a variety, with ring of polynomial functions R, and $f \in R$ be a polynomial function on V. We define*

$$U_f = \{v \in V : f(v) \neq 0\}$$

Proposition 2. With V, and U_f, we can give this the structure of a variety, whose ring of polynomial functions is $R[f^{-1}]$, the ring obtained by adding the inverse of f to R.

Example 8. *So for instance, if we looked at the variety $\mathbb{C}$, whose ring of coordinates in $\mathbb{C}[X]$, we could take X, a polynomial function on C. The vanishing locus of this is $\mathbb{C}\backslash\{0\}$.*

3.3 Counting Holes

Now finally, we can start counting holes. The ring whose spectrum I want to count holes in is going to be $k[X, Y, Y^{-1}, X^{-1}]$. I said that every finitely generated k-algebra could be realised as the coordinate ring of a variety; As a variety, we can think of this corresponding to $k^2\backslash\{0\}$. We want to repeat the procedure we performed for Čech Cohomology; that was:

1. Choose a nice open covering of our topological space
2. Make a chain complex, by looking at some nice breed of function from the open sets
3. Take cohomology!

This all looks fine, apart from one thing: what does it mean for our covering to be nice? Well, we'll just say that the affine opens are our nice opens. Now for lack of time I can't write it out, but doing the analogous process to what we did before w.r.t continuous functions, and learn that, roughly:

$$H^i_{(x,y)}(k[x,y]) = \begin{cases} 0 \text{ when } i \neq 2 \\ k^{\mathbb{Z}} \text{ when } i = 2 \end{cases}$$

Disclaimer 1. There is a beautiful theory behind this with gives some incredibly equivalences - unfortunately this means that the complex we make has an extra term at the beginning, which slightly screws up the dimension of our holes

Really what we've calculated here, is the local cohomology of $k[x, y]$ with support in (x, y), which is written $H^{\star}_{(x,y)}(k[x, y])$. So we have managed to count holes in a ring, and in fact you can do this more generally. Your ring doesn't need to come from an algebraic variety at all! To work out local cohomology, you need a choice of ring, and of ideal. We have the following nice fact

Proposition 3. If R is a ring with only one maximal ideal, m, then

$$\dim(R) = \sup_n(H^n_m(R) \neq 0)$$

and very quickly, for all the homological algebra fans out there, it turns out that

$$H^j_{\mathfrak{a}}(R) = \varinjlim \operatorname{Ext}(R/\mathfrak{a}^t, R)$$